Cardinalities and Cardinals (cardinal numbers)

Cantor: „Mächtigkeit" oder „Cardinalzahl" von M nennen wir den Allgemeinbegriff, welcher mit Hilfe unseres activen Denkvermögens dadurch aus der Menge M hervorgeht, dass von der Beschaffenheit ihrer verschiedenen Elemente m und von der Ordnung ihres Gegebenseins abstrahirt wird.

Das Resultat dieses zweifachen Abstractionsacts, die Cardinalzahl oder Mächtigkeit von M, bezeichnen wir mit
$\overline{\overline{M}}$

Cantor explains that this transforms (projects) M faithfully on a set of units (Einsen).

He then proves: sets have the same power iff there is a bijection between them:

① $M \propto \overline{\overline{M}} = \overline{\overline{N}} \approx N$

② Two sets with a bijection between them produce the same set of units.

When Cantor proved equalities he constructed bijections so that is what we adopt

$M \propto N$ means there is a bijection $f: M \to N$

M and N are equivalent

Comparing powers

If
- there is no subset of M that is equivalent to N, and
- there is a subset N_1 of N that is equivalent to M

Then we say $\overline{\overline{M}}$ is smaller than $\overline{\overline{N}}$

in symbols $\overline{\overline{M}} < \overline{\overline{N}}$

Note: always at most one of the possibilities

$\overline{\overline{M}} = \overline{\overline{N}}$, $\overline{\overline{M}} < \overline{\overline{N}}$, $\overline{\overline{N}} < \overline{\overline{M}}$

will occur

Cantor promised: later we'll see at least one will occur, in each case.

Nowadays: $M \preceq N$ means:

there is an injective map

$f : M \to N$.

We have two versions of less-than-or-equal

We : there is an injection $\overline{\overline{M}} \to \overline{\overline{N}}$

Cantor: (there is an injection $M \to N$, but no inj $N \to M$)

or (there is a bijection)

Cantor "$\overline{\overline{M}} \leq \overline{\overline{N}}$ and $\overline{\overline{N}} \leq \overline{\overline{M}}$" implies $\overline{\overline{M}} = \overline{\overline{N}}$

is easy, using simple logic

- $(A \wedge \neg C) \vee B$ $\quad\quad \overline{\overline{M}} \leq \overline{\overline{N}}$
- $(C \wedge \neg A) \vee B$ $\quad\quad \overline{\overline{N}} \leq \overline{\overline{M}}$

gives $(B \wedge B) \vee (B \wedge (C \wedge \neg A) \wedge (C \wedge \neg A)) \vee (A \wedge \neg C \wedge C \wedge \neg A)$

and this is just B.

But we have $B \to A \wedge B \to C$ and or $(B \vee \neg A) \wedge (B \vee \neg C)$

that will leave us with just B.

We have a more difficult task: $A \wedge C \to B$

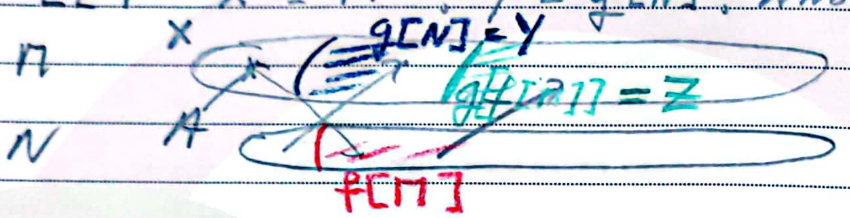
If there are injections $f: M \to N$ and $g: N \to M$ then there is a bijection $b: M \to N$.

Theorem [Dedekind-Cantor-Schroeder-Bernstein]

This is true

Proof Let $f: M \hookrightarrow N$ and $g: N \hookrightarrow M$ be given

Let $X = M$, $Y = g[N]$, and $Z = g[f[M]]$



So we have $X \supseteq Y \supseteq Z$ and

a bijection $h: X \to Z$: $h = f \circ g$.

Dedekind Let $A = X \setminus Y$

Let $\mathcal{B} = \{B \subseteq X : A \subseteq B \wedge h[B] \subseteq B\}$

Familiar arguments:

- $\mathcal{B} \neq \emptyset$ because $X \in \mathcal{B}$
- If $\mathcal{B}' \subseteq \mathcal{B}$ then $\bigcap \mathcal{B}' \in \mathcal{B}$

Let $C = \bigcap \mathcal{B}'$

- $A \subseteq B$ for all B so $A \subseteq C$.
- $h[C] = h[\bigcap\{B : B \in \mathcal{B}'\}]$ $\quad$ $C \subseteq B$ gives

$\subseteq \bigcap\{h[B] : B \in \mathcal{B}'\}$ $\quad$ $h[C] \subseteq h[B]$

$\subseteq \bigcap\{B : B \in \mathcal{B}'\}$

$= C$.

Let $W = \bigcap \mathcal{B}$, <u>the</u> <u>smallest</u> member of $\mathcal{B}$

Define $$b(x) = \begin{cases} h(x) & x \in W \\ x & x \in X \setminus W. \end{cases}$$

Then b is a bijection from X to Y.

- If $x \neq y$ in W then $h(x) \neq h(y)$
- If $x \neq y$ in $X \setminus W$ then $x \neq y$
- If $x \in W$ and $y \in X \setminus W$ then $h(x) \in W$ so $h(x) \neq y$

- Let $y \in Y$
  If $y \in Y \setminus W$ then $y = b(y)$
  If $y \in W$ then $W \setminus \{y\} \notin \mathcal{B}$
  So $A \not\subseteq W \setminus \{y\}$ or $h[W \setminus \{y\}] \not\subseteq W \setminus \{y\}$
  As $A \cap Y = \emptyset$ we have $A \subseteq W \setminus \{y\}$
  Hence there is a $w \in W \setminus \{y\}$
  such that $h(w) \notin W \setminus \{y\}$
  But $h(w) \in W$, so $h(w) = y$.

Both notion of less-than-or-equal are the same.

So $M \preccurlyeq N$: there is an injection $M \to N$
$M \approx N$: there is a bijection $M \leftrightarrow N$
$M \prec N$: there is an injection $M \to N$
but <u>no</u> injection $N \to M$.

Very often you see those written as
$|M| \leq |N|$, $|M| = |N|$, and $|M| < |N|$.
And expressed as
The cardinality of M is less than or equal to (or equal to, or less than) the cardinality of N.
Where "the cardinality" is essentially undefined.

But we shall <u>define</u> $|A|$ for <u>sets</u> that can be <u>well-ordered</u>:

$$|A| = \min\{\alpha : \alpha \approx A\}$$

SO $|A|$ IS AN ORDINAL AND A SPECIAL ONE AT THAT

α IS A CARDINAL (NUMBER) IF

$$\alpha = |\alpha|$$

SO α IS A CARDINAL IF

$$(\forall \beta < \alpha)(\beta \not\approx \alpha)$$

CARDINALS ARE ALSO CALLED INITIAL ORDINALS.

NOTATIONAL CONVENTION:

κ, λ, μ, AND ν TEND TO BE CARDINALS.

EXAMPLE: IF X IS A SET THEN $\aleph(X)$ IS A CARDINAL.

LEMMA a) IF $n \in \omega$ THEN $n \not\approx n+1$

b) IF $n \in \omega$ AND $\alpha \approx n$ THEN $\alpha = n$.

PROOF a) INDUCTION

b) USE a) AND D-C-S-B THEOREM.

SO EVERY NATURAL NUMBER IS A CARDINAL, AS IS ω.

DEFINITION:

A IS FINITE: $|A| < \omega$

A IS COUNTABLE: $|A| \leq \omega$

INFINITE = NOT FINITE

UNCOUNTABLE = NOT COUNTABLE.

ARITHMETIC LET κ AND λ BE CARDINALS

THEN $\kappa \oplus \lambda = |\kappa \times \{0\} \cup \lambda \times \{1\}|$

$\kappa \otimes \lambda = |\kappa \times \lambda|$

NOTE BOTH $\kappa \times \{0\} \cup \lambda \times \{1\}$ AND $\kappa \times \lambda$ ARE WELL-ORDERABLE.

- $\omega \oplus 1 = |\omega + 1| = |1 + \omega| = \omega < \omega + 1$ } NOT CARDINALS
- $\omega \otimes 2 = |\omega \cdot 2| = |2 \cdot \omega| = \omega < \omega \cdot 2$
- $n \oplus m = n + m$ $n \otimes m = n \cdot m$ FOR $n, m \in \omega$

  AND $n + m < \omega$ AND $n \cdot m < \omega$

INFINITE CARDINALS ARE LIMIT ORDINALS:

$1 + \alpha = \alpha$ WHEN $\alpha \geq \omega$

HENCE $|\alpha + 1| = |1 + \alpha| = |\alpha|$

THEOREM [HESSENBERG] IF κ IS AN INFINITE CARDINAL THEN $\kappa \oplus \kappa = \kappa \otimes \kappa = \kappa$.

PROOF [INDUCTION]

DEFINE $\langle \alpha, \beta \rangle \lhd \langle \gamma, \delta \rangle$

IF $\max\{\alpha, \beta\} < \max\{\gamma, \delta\}$ OR

$\max\{\alpha, \beta\} = \max\{\gamma, \delta\}$ AND $\alpha < \gamma$ OR

$\max\{\alpha, \beta\} = \max\{\gamma, \delta\}$ AND $\alpha = \gamma$ AND $\beta < \delta$.

THIS IS A WELL-ORDER OF On. (EXERCISE)

- ALWAYS: $\{\langle \gamma, \delta \rangle : \langle \gamma, \delta \rangle \lhd \langle 0, \alpha \rangle\} = \alpha \times \alpha$
- $\Gamma(\alpha)$ = ORDER-TYPE OF $\alpha \times \alpha$
- Γ IS A NORMAL FUNCTION
  - $\Gamma(\alpha + 1) = \Gamma(\alpha) + \alpha + \alpha + 1$ SO $\Gamma(\alpha) < \Gamma(\alpha + 1)$
  - IF α IS A LIMIT THEN

$$\alpha \times \alpha = \bigcup_{\beta < \alpha} (\beta \times \beta)$$

AND SO $\Gamma(\alpha) = \bigcup_{\beta < \alpha} \Gamma(\beta)$

$$\Big[\Gamma(\alpha) = \{ \mathrm{OT}(\mathrm{PRED}(\langle \gamma, \delta \rangle, \lhd)) : \langle \gamma, \delta \rangle \lhd \langle 0, \alpha \rangle \}$$

$$= \bigcup_{\beta < \alpha} \{ \mathrm{OT}(\mathrm{PRED}(\langle \gamma, \delta \rangle, \lhd)) : \langle \gamma, \delta \rangle \lhd \langle 0, \beta \rangle \} \Big]$$

- So $\Gamma(\omega) = \sup_{n\in\omega} \Gamma(n) = \sup_{n\in\omega} n^2 = \omega$
- NOTE $\{\langle\alpha,0\rangle : \alpha < \kappa\}$ HAS ORDERTYPE κ
  SO CERTAINLY $\Gamma(\kappa) \geq \kappa$.
- ASSUME $\Gamma(\lambda) = \lambda$ FOR ALL CARDINALS $\lambda < \kappa$.
  LET $\alpha < \kappa$, SO $|\alpha| \leq \alpha < \kappa$.
  THEN $|\alpha \times \alpha| = |\alpha| \otimes |\alpha| = |\alpha|$ (INDUCTIVE HYP)
  HENCE $\Gamma(\alpha) < \kappa$.
  BUT THEN $\Gamma(\kappa) = \bigcup_{\alpha<\kappa} \Gamma(\alpha) \leq \kappa$.
  WE FIND $\Gamma(\kappa) = \kappa$.

SO EASY ARITHMETIC: IF κ AND λ ARE INFINITE
THEN - $\kappa \oplus \lambda = \kappa \otimes \lambda = \max(\kappa, \lambda)$
- $|\kappa^{<\omega}| = \kappa$

INDUCTION: $\kappa^n \otimes \kappa = \kappa$

SO $|\kappa^{<\omega}| = \omega \otimes \kappa = \kappa$.

JUST TO BE SURE
AXIOM OF POWER SET $(\forall x)(\exists y)(\forall z)(z \subseteq x \to z \in y)$

HOMEWORK: WE HAVE $\aleph$ HARTOGS' ALEPH
- $\aleph(\omega)$ IS AN UNCOUNTABLE CARDINAL
  IN FACT IT IS THE FIRST
- FOR EVERY κ WE GET THE NEXT
  LARGEST CARDINAL AS $\aleph(\kappa)$; NOTATION κ^+

HERE COME THE ALEPHS:

$\omega_0 = \aleph_0$ FIRST INFINITE CARDINAL

$\omega_1 = \aleph_1 = \aleph(\omega)$

$\omega_{\alpha+1} = \aleph_{\alpha+1} = \aleph(\omega_\alpha)$

$\omega_\alpha = \aleph_\alpha = \sup_{\beta<\alpha} \omega_\beta$ IF α IS A LIMIT

THE SEQUENCES $\langle \omega_\alpha : \alpha \in On \rangle$ AND $\langle \aleph_\alpha : \alpha \in On \rangle$ EXIST BY THE RECURSION PRINCIPLE.

WHY TWO SYMBOLS

ω_α ORDINAL CONTEXT - SET OF ORDINALS

$\aleph_\alpha$ CARDINAL CONTEXT - LABEL TO DENOTE "THE NUMBER OF ELEMENTS"

CANTOR'S THEOREM [ONE OF MANY]

FOR ALL x WE HAVE $x \prec \mathcal{P}(x)$

- $a \mapsto \{a\}$ IS AN INJECTION
- IF $f: x \to \mathcal{P}(x)$ IS ANY MAP
  LET $R_f = \{a \in x : a \notin f(a)\}$
  THEN $R_f \notin \operatorname{RAN} f$.
  SO NO SURJECTION, HENCE NO BIJECTION

MANY RESULTS ABOUT CARDINALS USE AC

- $|X| \leq |Y|$ IFF THERE IS A SURJECTION $f: Y \to X$.
- IF $\kappa \geq \omega$ AND $|X_\alpha| \leq \kappa$ FOR $\alpha < \kappa$
  THEN $|\bigcup_{\alpha<\kappa} X_\alpha| \leq \kappa$
- [LÖWENHEIM-SKOLEM] LET $\kappa \geq \omega$
  LET $\mathcal{F}$ BE A SET OF FUNCTIONS $f: A^{n_f} \to A$ $n_f \in \omega$
  LET $B \subseteq A$, AND ASSUME $|B| \leq \kappa$ AND $|\mathcal{F}| \leq \kappa$
  THEN THE CLOSURE C OF B UNDER $\mathcal{F}$ SATISFIES $|C| \leq \kappa$.
  - $C_0 = B$, $C_{m+1} = \bigcup \{f[C_m^{n_f}] : f \in \mathcal{F}\}$, $C_\omega = \bigcup_{m\in\omega} C_m$
  - INDUCTION: $|C_m| \leq \kappa$ FOR ALL m
  - $|C_\omega| = |\bigcup_{m\in\omega} C_m| \leq \kappa$.
  - $C = C_\omega$ OF COURSE.