More cardinal exponentiation

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But first: cofinality; regular and singular cardinals.

- Let α and β be ordinals

  $f : \alpha \to \beta$ is cofinal if $\operatorname{ran} f$ is unbounded in β: $(\forall \eta < \alpha)(\exists \gamma < \beta)(\eta \leq f(\gamma))$

  The cofinality of β, $\operatorname{cf}(\beta)$, is the smallest α that admits a cofinal map into β.

- Clearly $\operatorname{cf}(\beta) \leq \beta$: use $\operatorname{id} : \beta \to \beta$
- If β is a successor then $\operatorname{cf}(\beta) = 1$
- If β is a limit then $\operatorname{cf}(\beta) \geq \omega$
- $\operatorname{cf}(\omega) = \omega$.

  Prove by induction: if $n < \omega$ and $f : n \to \omega$ then there is an $m < \omega$ such that $f[n] \subseteq m$.

- There is a strictly increasing cofinal map from $\operatorname{cf}(\beta)$ to β

  Let $g : \operatorname{cf}(\beta) \to \beta$ be cofinal
  - If $\gamma < \operatorname{cf}(\beta)$ then $g \restriction \gamma$ is not cofinal so $\sup g[\gamma] < \beta$
  - Recursion:

    $$f(\eta) = \sup\big(g[\eta+1] \cup \overbrace{\{f(\gamma)+1 : \gamma < \eta\}}^{\text{also not cofinal}}\big)$$

    f is strictly increasing by definition and cofinal because $g(\eta) \leq f(\eta)$ always

- There is even a continuous, strictly increasing, and cofinal map.

  [For limit ordinals γ replace $f(\gamma)$ by $\sup\{f(\delta) : \delta < \gamma\}$

- If α is a limit and $f : \alpha \to \beta$ is strictly increasing and cofinal then $\operatorname{cf}(\alpha) = \operatorname{cf}(\beta)$
  - $\operatorname{cf}(\beta) \leq \operatorname{cf}(\alpha)$ if $g : \operatorname{cf}(\alpha) \to \alpha$ is strictly increasing and cofinal then so is $f \circ g : \operatorname{cf}(\alpha) \to \beta$
  - $\operatorname{cf}(\alpha) \leq \operatorname{cf}(\beta)$ let $g : \operatorname{cf}(\beta) \to \beta$ be strictly increasing and cofinal

    Define $h : \operatorname{cf}(\beta) \to \alpha$ by

    $$h(\eta) = \min\{\gamma : f(\gamma) > g(\eta)\}$$

    Then h is cofinal.

So $cf(cf(\beta)) = cf(\beta)$

DEFINITION

β IS REGULAR IF $cf(\beta) = \beta$
AND β IS A LIMIT ORDINAL

- So $cf(\beta)$ IS ALWAYS REGULAR
- REGULAR ORDINALS ARE CARDINALS
- ω IS REGULAR
- AC : κ^+ IS ALWAYS REGULAR

  IF $f : \alpha \to \kappa^+$ IS COFINAL
  THEN $\kappa^+ = \bigcup\{f(\gamma) : \gamma < \alpha\}$
  WE HAVE $|f(\gamma)| \leq \kappa$ FOR ALL γ
  SO $\kappa^+ \leq \kappa \cdot |\alpha| = \max\{\kappa, |\alpha|\}$
  WE MUST HAVE $|\alpha| = \kappa^+$.
- $cf(\aleph_\alpha) = cf(\alpha)$ FOR ALL LIMIT ORDINALS α

A CARDINAL κ IS <u>SINGULAR</u> IF $cf(\kappa) < \kappa$.
So $\aleph_\omega$ IS SINGULAR.
SINCE WE ASSUME AC WE KNOW $\aleph_1, \aleph_2, \ldots$
ARE REGULAR.
BUT THERE ARE MODELS OF ZF WITH $cf(\omega_1) = \omega$.

So SINGULAR CARDINALS ARE LIMIT CARDINALS
WHAT ABOUT REGULAR LIMITS?
REGULAR LIMIT: WEAKLY INACCESSIBLE
REGULAR <u>STRONG LIMIT</u>: STRONGLY INACCESSIBLE
IF $\lambda < \kappa$ THEN $2^\lambda < \kappa$

UNDER GCH IT IS THE SAME THING.
BUT $2^{\aleph_0}$ CAN BE WEAKLY INACCESSIBLE.
HOWEVER...
IN ZFC + THERE IS A WEAKLY INACCESSIBLE
YOU CAN PROVE ZFC CONSISTENT, SO ...

WE'LL GET BACK TO REGULAR AND
SINGULAR CARDINALS LATER

REMEMBER: IF $\lambda \geq \aleph_0$ AND $2 \leq \kappa \leq 2^\lambda$
THEN $\kappa^\lambda = 2^\lambda$.
SO WHAT CAN WE SAY IF $2^\lambda < \kappa$??
HERE IS A FIRST BOUNDARY:

THEOREM [KÖNIG]
IF κ IS INFINITE AND $\lambda \geq cf(\kappa)$ THEN $\kappa^\lambda > \kappa$.

LET $f: cf(\kappa) \to \kappa$ BE COFINAL
LET $g: \kappa \to {}^{cf(\kappa)}\kappa$ BE ARBITRARY
DEFINE $h: cf(\kappa) \to \kappa$ BY
$$h(\alpha) = \min \kappa \setminus \{ g(\beta)(\alpha) : \beta < f(\alpha) \}$$
THEN $h \neq g(\beta)$ FOR ALL $\beta < \kappa$.
SO g IS NOT SURJECTIVE.

COROLLARY (BY CONTRAPOSITION)
$(2^\lambda)^\lambda = 2^\lambda$ SO $\lambda < cf(2^\lambda)$
CONCLUSION $2^{\aleph_0} \neq \aleph_\omega$

THIS IS ENOUGH TO SHOW THAT LIFE IS EASY IF WE ASSUME GCH.

- IF $\kappa \leq \lambda$ THEN $\kappa^\lambda = 2^\lambda = \lambda^+$
- IF $cf(\kappa) \leq \lambda < \kappa$ THEN $\kappa^\lambda = \kappa^+$
  FOR $\kappa^\lambda \geq \kappa^+$ AND $\kappa^\lambda \leq \kappa^\kappa = 2^\kappa = \kappa^+$
- IF $\lambda < cf(\kappa)$ THEN $\kappa^\lambda = \kappa$
  FOR ${}^\lambda\kappa = \bigcup \{ {}^\lambda\alpha : \alpha < \kappa \}$ (EVERY $f: \lambda \to \kappa$ IS BOUNDED)
  AND $|{}^\lambda\alpha| \leq \max(|\alpha|, \lambda)^+ \leq \kappa$ FOR ALL α
  AND SO $\kappa^\lambda \leq \kappa \cdot \kappa = \kappa$.

SO... WHAT CAN WE SAY IN GENERAL?
FIRST: SUMS AND PRODUCTS

$$\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} \{i\} \times \kappa_i \right|$$

$$\underbrace{\prod_{i \in I} \kappa_i}_{\text{CARDINAL}} = \left| \underbrace{\prod_{i \in I} \kappa_i}_{\text{SET}} \right|$$

IN GENERAL :

IF λ IS INFINITE AND $\kappa_i > 0$ FOR ALL $i < \lambda$

THEN $\sum_{i<\lambda} \kappa_i = \lambda \otimes \sup_{i<\lambda} \kappa_i$

LET $\kappa = \sup_{i<\lambda} \kappa_i$

① $\bigcup_{i<\lambda} \{i\} \times \kappa_i \leq \lambda \times \kappa$, WHICH GIVES $\leq$

② $\lambda \times \{0\} \leq \bigcup_{i<\lambda} \{i\} \times \kappa_i$

$\{j\} \times \kappa_j \leq \bigcup_{i<\lambda} \{i\} \times \kappa_i$ FOR ALL j

SO $\lambda, \kappa \leq \sum_{i<\lambda} \kappa_i$, HENCE $\lambda \otimes \kappa \leq \sum_{i\in I} \kappa_i$

So: IF $\lambda \leq \sup_{i<\lambda} \kappa_i$ THEN $\sum_{i<\lambda} \kappa_i = \sup_{i<\lambda} \kappa_i$

AND :

κ IS SINGULAR IFF $\kappa = \sum_{i<\lambda} \kappa_i$ WITH $\lambda < \kappa$ AND $\kappa_i < \kappa$ FOR ALL i.

EXERCISE : FORMULATE AND VERIFY (GENERAL) COMMUTATIVITY AND ASSOCIATIVITY PROPERTIES OF INFINITE SUMS AND PRODUCTS, AS WELL AS RULES FOR DISTRIBUTIVITY-LIKE FORMULAS, SAY $\lambda^{\sum_i \kappa_i} = \prod_i (\lambda^{\kappa_i})$ ETC.

IF $\kappa_i \geq 2$ FOR ALL i THEN $\sum_{i\in I} \kappa_i \leq \prod_{i\in I} \kappa_i$

(IF $|I| \geq 2$ THEN $\sum_{i\in I} 1 > \prod_{i\in I} 1$.)

MAP $\bigcup_{i\in I} \{i\} \times \kappa_i$ INTO $\prod_{i\in I} \kappa_i$

- $|I| = 2$: $\{0\} \times \kappa_0 \cup \{1\} \times \kappa_1$
- $|I| \geq 3$: 〃

IF $\alpha > 0$ THEN $\langle i, \alpha \rangle \longmapsto f_{i\alpha} : \begin{matrix} j \mapsto 0 & j \neq i \\ i \mapsto \alpha & \end{matrix}$

$\langle i, 0 \rangle \longmapsto g_i : \begin{matrix} j \mapsto 1 & j \neq i \\ i \mapsto 0 & \end{matrix}$

IF λ IS INFINITE AND $\langle \kappa_i : i<\lambda \rangle$ IS A NON-DECREASING SEQUENCE OF NON-ZERO CARDINALS THEN

$$\prod_{i<\lambda} \kappa_i = (\sup_{i<\lambda} \kappa_i)^\lambda$$

LET $\kappa = \sup_{i<\lambda} \kappa_i$.

SO $\prod_{i<\lambda} \kappa_i \subseteq {}^\lambda\kappa$ AS SETS

AND HENCE $\prod_{i<\lambda} \kappa_i \leq \kappa^\lambda$ AS CARDINALS

CONVERSELY VIA $\lambda \approx \lambda \times \lambda$ WE WRITE

$\lambda = \bigcup_{i<\lambda} A_i$ WITH $A_i \cap A_j = \emptyset$ IF $i \neq j$ AND $|A_i| = \lambda$ FOR ALL i

THEN $\kappa = \sup_{j \in A_i} \kappa_j$ FOR ALL i

ALSO $\prod_{j \in A_i} \kappa_j \geq \kappa_j$ FOR EACH $j \in A_i$

AND SO $\prod_{j \in A_i} \kappa_j \geq \kappa$ FOR ALL i

AND SO

$$\prod_{i<\lambda} \kappa_i = \prod_{i<\lambda} \Big(\prod_{j \in A_i} \kappa_j\Big) \geq \prod_{i<\lambda} \kappa = \kappa^\lambda.$$

KÖNIG'S GENERAL INEQUALITY.

IF $\kappa_i < \lambda_i$ FOR ALL $i \in I$

THEN $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$

LET $f : \bigcup_{i \in I} \{i\} \times \kappa_i \longrightarrow \prod_{i \in I} \lambda_i$ (SET) BE A MAP.

FOR EACH i CONSIDER THE COMPOSITION $\pi_i \circ f$

AND RESTRICT IT TO $\{i\} \times \kappa_i$ WE GET

$V_i = \{ \pi_i(f(i,\alpha)) : \alpha < \kappa_i \} \subseteq \lambda_i$

BECAUSE $\kappa_i < \lambda_i$ THE INCLUSION IS PROPER

DEFINE $g \in \prod_{i \in I} \lambda_i$ BY

$g(i) = \min \lambda_i \setminus V_i$

THEN $g \notin \operatorname{ran} f$ $\quad g(i) \notin \{ \pi_i(f(i,\alpha)) : \alpha < \kappa_i \}$

IMPLIES $g \notin f[\{i\} \times \kappa_i]$.

CONSEQUENCES:

- $\kappa < 2^\kappa$ $\quad \kappa_i = 1, \lambda_i = 2$
- $\kappa < \operatorname{cf}(2^\kappa)$ $\quad \kappa_i < 2^\kappa$ FOR $i < \kappa$:

  $\sum_{i<\kappa} \kappa_i < \prod_{i<\kappa} 2^\kappa = 2^{\kappa \otimes \kappa} = 2^\kappa$.
- $\kappa < \kappa^{\operatorname{cf}(\kappa)}$ $\quad \kappa = \sum_{i<\lambda} \kappa_i$ $\quad \lambda = \operatorname{cf} \kappa$, $\kappa_i < \kappa$ ALL i

  SO $\kappa = \sum_{i<\lambda} \kappa_i < \prod_{i<\lambda} \kappa = \kappa^{\operatorname{cf}(\kappa)}$

WHAT CAN WE SAY ABOUT $\kappa \mapsto 2^\kappa$?

- $\kappa < \lambda \to 2^\kappa \leq 2^\lambda$ $\quad [2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ IN COHEN'S MODEL FOR $\neg$CH$]$
- $\kappa < \mathrm{cf}\, 2^\kappa$
- IF κ IS A LIMIT CARDINAL THEN $2^\kappa = (2^{<\kappa})^{\mathrm{cf}\,\kappa}$.

NOW IN GENERAL $\kappa^{<\lambda} = \sup\{\kappa^\mu : \mu$ CARDINAL, $\mu < \lambda\}$

ASIDE: USEFUL FACT IF $\kappa \geq \lambda$ THEN
$\kappa^\lambda = |[\kappa]^\lambda|$ $\quad ([\kappa]^\lambda = \{A \subseteq \kappa : |A| = \lambda\})$
- ${}^\lambda\kappa \subseteq [\lambda \times \kappa]^\lambda \subseteq [\kappa \times \kappa]^\lambda$ SO $\kappa^\lambda \leq |[\kappa]^\lambda|$
- $[\kappa]^\lambda \subseteq \{\mathrm{Ran}\, f : f \in {}^\lambda\kappa\}$; $|[\kappa]^\lambda| \leq \kappa^\lambda$.

WRITE $\kappa = \Sigma_{i < \mathrm{cf}\,\kappa} \kappa_i$ WITH $\kappa_i < \kappa$ (ALL i)

$$2^\kappa = 2^{\Sigma_i \kappa_i} = \prod_i 2^{\kappa_i} \leq \prod_i 2^{<\kappa} = (2^{<\kappa})^{\mathrm{cf}\,\kappa} \leq (2^\kappa)^{\mathrm{cf}\,\kappa} \leq 2^\kappa.$$

FOR REGULAR CARDINALS

$\kappa < \lambda \to 2^\kappa \leq 2^\lambda$ AND $\kappa < \mathrm{cf}\, 2^\kappa$ $(\leq 2^\kappa)$

ARE THE ONLY RESTRICTIONS [EASTON].

FOR SINGULAR CARDINALS ......

- IF κ IS SINGULAR AND $\mu \mapsto 2^\mu$ IS CONSTANT ON AN INTERVAL $(\nu, \kappa]$ WITH VALUE λ $(\nu < \kappa)$ THEN $2^\kappa = \lambda$.

  TAKE $\nu \geq \mathrm{cf}\,\kappa$ SUCH THAT $2^\mu = \lambda$ IF $\nu \leq \mu < \kappa$.
  BUT THEN $2^{<\kappa} = 2^\nu = \lambda$ AND SO
  $$2^\kappa = (2^{<\kappa})^{\mathrm{cf}\,\kappa} = (2^\nu)^{\mathrm{cf}\,\kappa} = 2^\nu \; (= \lambda).$$
- USEFUL FUNCTION: GIMEL $\gimel(\kappa) = \kappa^{\mathrm{cf}\,\kappa}$.
- IF $\mu \mapsto 2^\mu$ IS NOT CONSTANT ON AN INTERVAL $[\nu, \kappa]$ THEN $2^{<\kappa} = \lim_{\alpha \to \kappa} 2^{|\alpha|}$ SO $\mathrm{cf}(2^{<\kappa}) = \mathrm{cf}(\kappa)$
  SO IF $\lambda = 2^{<\kappa}$ THEN $2^\kappa = \lambda^{\mathrm{cf}\,\kappa} = \lambda^{\mathrm{cf}\,\lambda} = \gimel(\lambda)$.

SUMMARY
- κ SUCCESSOR $2^\kappa = \kappa^\kappa = \kappa^{\mathrm{cf}\,\kappa} = \gimel(\kappa)$
- κ LIMIT AND $\mu \mapsto 2^\mu$ IS EVENTUALLY CONSTANT BELOW κ THEN
  $$2^\kappa = 2^{<\kappa} \cdot \gimel(\kappa)$$
- κ LIMIT AND $\mu \mapsto 2^\mu$ NOT EVENTUALLY CONSTANT BELOW κ THEN
  $$2^\kappa = \gimel(2^{<\kappa})$$

Exponentiation.

- If κ is regular and $\lambda < \kappa$
  then ${}^\lambda\kappa = \bigcup_{\alpha<\kappa} {}^\lambda\alpha$ [each map is bounded]
  and so $\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\kappa$. [via $\kappa \otimes \sup_{\alpha<\kappa} |\alpha|^\lambda$]
- Hausdorff's formula:
  $$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \otimes \aleph_{\alpha+1}.$$
- If κ is a limit and $\lambda \geq cf\kappa$ then
  $$\kappa^\lambda = (\operatorname{Lim}_{\alpha\to\kappa} \alpha^\lambda)^{cf\kappa}$$
  Say $\kappa = \sum_{i<cf\kappa} \kappa_i$ $(\kappa_i \cdot 2\kappa \cdot \text{all } i)$
  So $\kappa^\lambda \leq (\prod_{i<cf\kappa} \kappa_i)^\lambda = \prod_{i<cf\kappa} \kappa_i^\lambda \leq (\operatorname{Lim}_\kappa \alpha^\lambda)^{cf\kappa}$
  $\leq (\kappa^\lambda)^{cf\kappa} = \kappa^\lambda$.

Summary

Let λ be infinite for all infinite κ
we can compute κ^λ as follows

- $\kappa \leq 2^\lambda \to \kappa^\lambda = 2^\lambda$ [we know already]
- more generally if $\kappa \leq \mu^\lambda$ for some $\mu < \kappa$
  then $\kappa^\lambda = \mu^\lambda$. $[\mu^\lambda \leq \kappa^\lambda \leq (\mu^\lambda)^\lambda = \mu^\lambda]$
- if $\kappa > \lambda$ and $\mu^\lambda < \kappa$ for all $\mu < \kappa$
  then: if $cf\kappa > \lambda$ then $\kappa^\lambda = \kappa$
  if $cf\kappa \leq \lambda$ then $\kappa^\lambda = \kappa^{cf\kappa}$
  κ successor: $\kappa^\lambda = \mu^\lambda \cdot \kappa = \kappa$ if $\kappa = \mu^+$
  κ limit: $\operatorname{Lim}_{\alpha\to\kappa} \alpha^\lambda = \kappa$
  $\lambda < cf\kappa$ $\kappa^\lambda = \operatorname{Lim}_{\alpha\to\kappa} \alpha^\lambda = \kappa$
  $\lambda \geq cf\kappa$ $\kappa^\lambda = (\operatorname{Lim}_{\alpha\to\kappa} \alpha^\lambda)^{cf\kappa} = \kappa^{cf\kappa}$

So, also κ^λ can be reduced to the Gimel function.
The value of κ^λ is

- 2^λ ←-- already reduced to Gimel
- κ
- $\gimel(\mu)$ for some μ with $cf\mu \leq \lambda < \mu$.
  If $\kappa^\lambda > 2^\lambda \otimes \kappa$ take $\mu = \min\{\nu : \nu^\lambda = \kappa^\lambda\}$ $(<\kappa)$
  note if $\nu < \mu$ then $\nu^\lambda < \mu$
  then $\mu^\lambda = \mu^{cf\mu}$.

Most of the research on cardinal arithmetic concentrates on the Gimel function.
See Chapter 24 of Jech's Book.