THE CONSTRUCTIBLE UNIVERSE

GOAL: SHOW THAT THE CONJUNCTION

ZF + AC + GCH

IS CONSISTENT, ASSUMING ZF IS CONSISTENT.

HOW? BUILD A MODEL FOR ZF + AC + GCH, STARTING FROM A MODEL FOR ZF.

WHAT MODEL OF ZF? WE <u>ASSUME</u> $V = \bigcup_{\alpha \in ORD} V_\alpha$ IS OUR INITIAL MODEL.

HOW DO WE BUILD THAT SECOND MODEL THEN?

WE BUILD A (NARROWER) HIERARCHY $\langle L_\alpha : \alpha \in Ord \rangle$ AND LET $L = \bigcup_{\alpha \in Ord} L_\alpha$ BE THE MODEL.

RATHER THAN SIMPLY ADDING ALL SUBSETS, LIKE WE DID WHEN BUILDING $\langle V_\alpha : \alpha \in Ord \rangle$, WE SHALL BE MORE PARSIMONIOUS (FRUGAL, THRIFTY?) $L_{\alpha+1}$ WILL CONSIST OF ALL SETS <u>DEFINABLE</u> OVER L_α (OR: FROM L_α AND ITS ELEMENTS).

SO WE NEED TO DEFINE WHAT <u>DEFINABLE</u> MEANS, AND FOR THAT WE MUST KNOW WHAT IT MEANS TO RELATIVIZE A FORMULA.

FOR, IN GENERAL A SUBSET OR ELEMENT OF A STRUCTURE M FOR SOME LANGUAGE $\mathcal{L}$ IS DEFINABLE IF IT IS DESCRIBED BY A FORMULA.

IN A GROUP: $x = e$ IFF $G \models \varphi(x)$, WHERE $\varphi(x)$ IS $(\forall y)(x * y = y)$

BUT $G \models \varphi(x)$ MEANS $(\forall y \in G)(x * y = y)$

WE MUST BOUND THE QUANTIFIERS BY THE DOMAIN OF THE MODEL.

SO, IF M IS A SET, OR A CLASS, THEN FOR EVERY FORMULA φ WE DEFINE ITS RELATIVATION φ^M TO M RECURSIVELY

- $(x \in y)^M$ IS $x \in y$
- $(x = y)^M$ IS $x = y$
- $(\varphi \wedge \psi)^M$ IS $\varphi^M \wedge \psi^M$, DITTO FOR $\neg, \vee, \to, \leftrightarrow$
- $((\forall x)\varphi)$ IS $(\forall x \in M)\varphi^M$ ... FORMALLY $(\forall x)(x \in M \to \varphi^M)$
- $((\exists x)\varphi)$ IS $(\exists x \in M)\varphi^M$. $(\exists x)(x \in M \wedge \varphi^M)$

WE SAY $A \subseteq M$ IS <u>DEFINABLE</u> OVER M, OR $(M, \in)$, IF THERE ARE A FORMULA $\varphi(x, y_1, \dots, y_k)$ AND ELEMENTS $m_1, \dots, m_k$ IN M SUCH THAT

$$A = \{x \in M : \varphi^M(x, m_1, \dots, m_k)\}$$
$$(\{x \in M : (M, \in) \models \varphi[x, m_1, \dots, m_k]\})$$

IN GENERAL

$$\mathrm{DEF}(M) = \{A \subseteq M : A \text{ IS DEFINABLE OVER } (M, \in)\}$$

- $M \in \mathrm{DEF}(M)$: $M = \{x \in M : (x = x)^M\}$
- $M \subseteq \mathrm{DEF}(M)$ IF M IS TRANSITIVE : $x = \{y \in M : (y \in x)^M\}$
- $\mathrm{DEF}(M) \subseteq \mathcal{P}(M)$.

THE HIERARCHY $\langle L_\alpha : \alpha \in \mathrm{ORD}\rangle$

- $L_0 = \emptyset$
- $L_{\alpha+1} = \mathrm{DEF}(L_\alpha)$
- $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ (α A LIMIT)
- $L = \bigcup_{\alpha \in \mathrm{ORD}} L_\alpha$

FOR ALL α WE HAVE
- L_α IS TRANSITIVE
- $\alpha = L_\alpha \cap \mathrm{ORD}$
- IF $\beta < \alpha$ THEN $L_\beta \in L_\alpha$

INDUCTION ON α:

$\alpha = 0$ CLEAR

α LIMIT ALSO CLEAR BY TAKING UNIONS

$\alpha = \beta + 1$:- SINCE $L_\beta \subseteq L_\alpha \subseteq \mathcal{P}(L_\beta)$,
EVERY MEMBER OF L_α IS A SUBSET OF L_β
- WE HAVE $\beta = L_\beta \cap \mathrm{ORD} \subseteq L_\alpha \cap \mathrm{ORD}$
  $\beta = \{x \in L_\beta : x \text{ IS TRANSITIVE AND } \underline{\text{LINEARLY}} \text{ ORDERED BY } \in\}$
  - ALL QUANTIFIERS ARE RESTRICTED BY x HENCE BY L_β
  AND SO $\beta \in \mathrm{DEF}(L_\beta) = L_\alpha$
  SO $\alpha \subseteq L_\alpha \cap \mathrm{ORD}$
  BUT $\alpha \notin \mathcal{P}(L_\beta)$ SO $\alpha = L_\alpha \cap \mathrm{ORD}$.
- L_β IS TRANSITIVE, SO $L_\beta \in \mathrm{DEF}(L_\beta)$
  AND $\gamma < \beta \to L_\gamma \in L_\beta \in L_\alpha$
  SO $L_\gamma \in L_\alpha$

WE ALSO HAVE A RANK:

$$\rho(x) = \min\{\beta : x \in L_{\beta+1}\}$$

AND $L_\alpha = \{x \in L : \rho(x) < \alpha\}$

- $L_\alpha \subseteq V_\alpha$ FOR ALL α. BY INDUCTION
- IF $F \subseteq L_\alpha$ IS FINITE THEN $F \in L_{\alpha+1}$

  LET $n \in \omega$ AND LET $f : n \to F$ BE A BIJECTION
  - $f[\emptyset] = \emptyset \in L_{\alpha+1}$
  - ASSUME $f[i] \in L_{\alpha+1}$ THEN $f[i+1] = f[i] \cup \{f(i)\}$
    - $\{f(i)\} \in L_{\alpha+1}$ IT IS $\{x \in L_\alpha : (x = f(i))^{L_\alpha}\}$
    - LET φ AND $m_1, \dots, m_n$ BE SUCH THAT

      $f[i] = \{x \in L_\alpha : \varphi^{L_\alpha}(x, m_1, \dots, m_n)\}$
    - $f[i+1] = \{x \in L_\alpha : \varphi^{L_\alpha}(x, m_1, \dots, m_n) \vee (x = f(i))^{L_\alpha}\}$

  THEN $F = f[n] \in L_{\alpha+1}$.
- FOR $n \in \omega$ $L_n = V_n$ [INDUCTION]
- $L_\omega = V_\omega$ [UNION]
- (AC) FOR $\alpha \geq \omega$ $|L_\alpha| = |\alpha|$

  CERTAINLY $|\alpha| \leq |L_\alpha|$ AS $\alpha \subseteq L_\alpha$
  - BY ASSUMPTION, IF $\beta < \alpha$ THEN $|L_\beta| < \aleph_0 \leq |\alpha|$

    OR $|L_\beta| = |\beta| \leq |\alpha|$

    SO IF α IS A LIMIT THEN, BY AC,

    $|L_\alpha| \leq |\alpha| \cdot |\alpha| = |\alpha|$.

    IF $\alpha = \beta+1$ THEN $|L_\beta| = |\beta| = |\alpha|$

    BUT $|\mathrm{Def}(L_\beta)| = |\beta| = |\alpha|$

    BECAUSE WE HAVE COUNTABLY MANY FORMULAS AND $|\beta|$ MANY FINITE SUBSETS OF L_β, AGAIN BY AC.
- IN FACT AC IS NOT NEEDED AS WE SHALL SEE LATER

SO NOW: (ZF): L IS A MODEL OF ZF.

- L IS TRANSITIVE SO WE HAVE EXTENSIONALITY
- PAIRING: IF $x, y \in L_\alpha$ THEN $\{x, y\} \in L_{\alpha+1}$
- UNION: IF $x \in L_\alpha$ THEN $\bigcup x = \{y \in L_\alpha : (\exists z \in x)(y \in z)\}$

  BUT BY TRANSITIVITY

  $(\exists z \in x)(y \in z) \leftrightarrow [(\exists z \in x)(y \in z)]^{L_\alpha}$
- INFINITY: $\omega \in L_{\omega+1}$

- POWER SET AND REPLACEMENT

  THIS IS WHY WE ONLY ASKED FOR SETS THAT CONTAIN THE WANTED SETS.

  - LET $\alpha = \sup\{g(y) : y \in P(x) \cap L\}$ [SEPARATION IN ZF]

    SO IF $y \subseteq x$ AND $y \in L$ THEN $y \in L_{\alpha+1}$

    SO $L_{\alpha+1}$ SATISFIES

    $$[(\forall y)(y \subseteq x \rightarrow y \in z)]^L$$

  - IF $A \in L$ AND ALSO $w_1, \dots, w_n \in L$ ARE SUCH THAT

    $\forall x \in A\ \exists! y \in L\ \varphi(x, y, A, w_1, \dots, w_n)$

    THEN TAKE $\alpha = \sup\{g(y)+1 : (\exists x \in A)\varphi^L(x, y, w_1, \dots, w_n)\}$

    THEN L_α IS AS REQUIRED BY REPLACEMENT.

- SEPARATION:

  LET $\varphi(x, z, w_1, \dots, w_n)$ BE A FORMULA WITH ITS FREE VARIABLES SHOWN.

  SO IF $z, w_1, \dots, w_n \in L$ THEN

  $$\{x \in z : \varphi^L(x, z, w_1, \dots, w_n)\}$$

  MUST BE IN L

  SO TAKE α WITH $z, w_1, \dots, w_n \in L_\alpha$

  BUT NOW: $\{x \in z : \varphi^{L_\alpha}(x, z, w_1, \dots, w_n)\} \in L_{\alpha+1}$

  HOWEVER $\{x \in z : \varphi^{L_\alpha}(x, z, w_1, \dots, w_n)\} \stackrel{?}{=} \{x \in z : \varphi^L(x, z, w_1, \dots, w_n)\}$

  NOT NECESSARILY:

  $$\forall y \in L_\alpha \text{ vs } \forall y \in L \ // \ \exists y \in L_\alpha \text{ vs } \exists y \in L$$

  WHAT WE USE HERE AND IN MANY OTHER PLACES IS THE FOLLOWING:

  $$(\exists \beta > \alpha)(\forall a, b, c_1, \dots, c_k \in L_\beta)(\varphi^L(a, b, \underline{c}) \leftrightarrow \varphi^{L_\beta}(a, b, \underline{c}))$$

  SO WITH THIS WE FIND

  $$\{x \in z : \varphi^L(x, z, w_1, \dots, w_n)\} = \{x \in z : \varphi^{L_\beta}(x, z, w_1, \dots, w_n)\} \in L_{\beta+1}$$

PLUS: THE DEFINITION OF $\mathrm{DEF}(M)$ IS NOT FORMALIZABLE IN ZF.

WE NEED TO HAVE A CLOSER LOOK AT DEFINABILITY.

AND TO SEE HOW WE GET $\varphi^L \leftrightarrow \varphi^{L_\beta}$

SIMPLE FORMULAS

A FORMULA IS A Δ_0-FORMULA IF IT IS

- WITHOUT QUANTIFIERS, OR
- OF THE FORM $\neg\varphi$, $\varphi\wedge\psi$, $\varphi\vee\psi$, $\varphi\to\psi$, $\varphi\leftrightarrow\psi$ WITH Δ_0-FORMULAS φ AND ψ, OR
- OF THE FORM $(\exists x\in y)\varphi$ OR $(\forall x\in y)\varphi$ WITH φ A Δ_0-FORMULA

WHY USEFUL?

① IF M IS TRANSITIVE AND φ IS A Δ_0-FORMULA THEN $ZF \vdash \varphi^M(x_1,\dots,x_m) \leftrightarrow \varphi(x_1,\dots,x_m)$ $(x_1,\dots,x_m\in M)$

IN THAT CASE φ IS SAID TO BE <u>ABSOLUTE</u> FOR M.

INDUCTION ON COMPLEXITY

- $(x\in y)^M$ IS $x\in y$ AND $(x=y)^M$ IS $x=y$
- $\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$ CLEAR
- IF $y\in M$ AND $(\exists x\in y)\varphi$
  THEN, AS $y\in M$, THERE IS AN $x\in M$
  WITH $x\in y\wedge\varphi$
  THEN ALSO $x\in y\wedge\varphi^M$, BECAUSE $\varphi^M\leftrightarrow\varphi$
  SO $(\exists x\in M)(x\in y\wedge\varphi^M)$
  WHICH IS $((\exists x)(x\in y\wedge\varphi))^M$
- IF $y\in M$ AND $((\exists x)(x\in y\wedge\varphi))^M$
  THEN $(\exists x\in M)(x\in y\wedge\varphi^M)$
  OR $(\exists x\in y)\varphi^M$
  AND SO $(\exists x\in y)\varphi$ BY $\varphi^M\leftrightarrow\varphi$

② MANY NOTIONS ARE EXPRESSED/EXPRESSABLE IN Δ_0-FORMULAS, IN ZF – "POWERSET".

SEE LEMMA 12.10 IN JECH'S BOOK

SEE THEOREM IV.3.9, IV.3.11, IV.5.1 … MORE EXPLICIT

$x\in y$, $x=y$, $x\subseteq y$, $\{x,y\}$, $\{x\}$, $\langle x,y\rangle$, $\emptyset$, $x\cup y$, $x\cap y$, $x\setminus y$, $S(x)$ $(=x\cup\{x\})$, $\bigcup x$, $\bigcap x$ $[\bigcap\emptyset=\emptyset]$

x IS TRANSITIVE, x IS AN ORDINAL [FOUNDATION!];

ALSO: LIMIT VERSUS SUCCESSOR

Absoluteness.

In general: if M is a set and N a set or class with $M \subseteq N$ then $\varphi(x_1,..,x_k)$ is absolute for M, N if for all $m_1,..,m_k \in M$ we have $\varphi^M(m_1,..,m_k) \leftrightarrow \varphi^N(m_1,..,m_k)$

And: $\varphi(x_1,..,x_k)$ is upward absolute for M, N
if $\varphi^M(m_1,..,m_k) \to \varphi^N(m_1,..,m_k)$ (all $m_1,..,m_k \in M$)

$\varphi(x_1,..,x_k)$ is downward absolute for M, N
if $\varphi^N(m_1,..,m_k) \to \varphi^M(m_1,..,m_k)$ (all $m_1,..,m_k \in M$)

[Remember we are looking for absoluteness for L_β, L voor many β.]

There is more absoluteness than just Δ_0
For example if M is transitive and satisfies $ZF-P$ (Why $ZF-P$? There are natural such M.) then the following are absolute for M, V.

- x is finite
- $x = A^n$
- $x = A^{<\omega}$
- "R well-orders A" and $\alpha = \text{ordertype}(A,R)$.

• x is finite iff $\exists f\, \varphi(x,f)$, where $\varphi(x,f)$ says f is a function $\wedge$ dom $f = x$ $\wedge$ ran $f \in \omega$ $\wedge$ f is 1-1
  - f is a function is Δ_0
  - dom $f = x$ is Δ_0
  - ran $f \in \omega$ is Δ_0
  - f is 1-1 is Δ_0

This is upward absolute for M, V
if $(\exists f \in M)\, \varphi^M(x,f)$ then $(\exists f \in M)\, \varphi(x,f)$
and then also $(\exists f)\, \varphi(x,f)$.

Also downward: if $x \in M$ and $\varphi(x,f)$ then $f \in M$ for f is a finite subset of M and hence a member of M
so $\exists f\, \varphi(x,f)$ implies $(\exists f \in M)\, \varphi(x,f)$

The rest: exercise or later

WE REDEFINE DEF SO THAT IT BECOMES ABSOLUTE FOR TRANSITIVE SETS THAT SATISFY ZF − P.

TO THIS END WE DEFINE TEN OPERATIONS

$G_1(X,Y) = \{X,Y\}$

$G_2(X,Y) = X \times Y$

$G_3(X,Y) = \in(X,Y) = \{\langle u,v\rangle : u \in X \wedge v \in Y \wedge u \in v\}$

$G_4(X,Y) = X \setminus Y$

$G_5(X,Y) = X \cap Y$

$G_6(X) = \bigcup X$

$G_7(X) = \operatorname{dom} X$

$G_8(X) = \{\langle u,v\rangle : \langle v,u\rangle \in X\}$

$G_9(X) = \{\langle u,v,w\rangle : \langle u,w,v\rangle \in X\}$

$G_{10}(X) = \{\langle u,v,w\rangle : \langle v,w,u\rangle \in X\}$

THESE ARE GÖDEL OPERATIONS.

THEOREM

IF $\varphi(u_1,\dots,u_m)$ IS A Δ_0-FORMULA THEN THERE IS A COMPOSITION G OF $G_1, G_2, \dots, G_{10}$ SUCH THAT FOR ALL SETS $X_1,\dots,X_m$ WE HAVE

$G(X_1,\dots,X_m) = \{\langle u_1,\dots,u_m\rangle : u_1 \in X_1 \wedge \dots \wedge u_m \in X_m \wedge \varphi(u_1,\dots,u_m)\}$

AND CONVERSELY IF G IS SUCH A COMBINATION THEN $Z = G(X_1,\dots,X_m)$ CAN BE EXPRESSED AS A Δ_0-FORMULA.

CONSEQUENCES:

IF M IS TRANSITIVE AND CLOSED UNDER THE GÖDEL OPERATION THEN M SATISFIES SEPARATION FOR Δ_0-FORMULAS

SAY $\varphi(u,p_1,\dots,p_m)$ IS Δ_0 AND $x, p_1,\dots,p_m \in M$

LET $y = \{u \in x;\ \varphi(u,p_1,\dots,p_m)\}$

WE MUST SHOW $y \in M$; TAKE G FOR φ SUCH THAT $G(X,\{p_1\},\dots,\{p_m\}) = \{\langle u,p_1,\dots,p_m\rangle : u \in X \wedge \varphi(u,p_1,\dots,p_m)\}$

SO $y = \{u : \exists u_1 \cdots \exists u_m\, \langle u,u_1,\dots,u_m\rangle \in G(X,\{p_1\},\dots,\{p_m\})\}$

$= \operatorname{DOM}^m G(X,\{p_1\},\dots,\{p_m\})$ $(= G_7^m(G(X,\{p_1\},\dots,\{p_m\})))$

ALSO $\{p_i\} = G_1(p_i,p_i)$ FOR ALL i SO WE ARE DONE.