· GÖDEL OPERATIONS ARE ABSOLUTE FOR TRANSITIVE SETS BECAUSE Δ_0-FORMULAS ARE.

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BUT FIRST: ABSOLUTENESS AND REFLECTION

LET M AND N BE CLASSES WITH $M \subseteq N$.
LET $\varphi_1, \ldots, \varphi_n$ BE A LIST OF FORMULAS THAT IS CLOSED UNDER TAKING SUBFORMULAS:
IF φ_i IS $\neg\psi$ THEN $\psi = \varphi_j$ FOR SOME j
IF φ_i IS $\psi \wedge \sigma$ THEN $\psi = \varphi_j$ AND $\sigma = \varphi_k$ FOR SOME j AND k
IF φ_i IS $(\exists x)\psi(x, y_1, \ldots, y_n)$ THEN $\psi = \varphi_j$ FOR SOME j.
THEN THE FOLLOWING ARE EQUIVALENT
a) $\varphi_1, \ldots, \varphi_n$ ARE ABSOLUTE FOR M, N
b) WHENEVER $\varphi_i(y_1, \ldots, y_k)$ IS $(\exists x)\varphi_j(x, y_1, \ldots, y_k)$ (WITH ALL FREE VARIABLES SHOWN) WE HAVE

$$\forall m_1, \ldots, m_k \in M \left(\exists x \in N\, \varphi_j^N(x, m_1, \ldots, m_k) \rightarrow \exists x \in M\, \varphi_j^N(x, m_1, \ldots, m_k)\right)$$

(a) → (b) TAKE $m_1, \ldots, m_k \in M$ ASSUME $(\exists x \in N)\, \varphi_j^N(x, m_1, \ldots, m_k)$
THIS MEANS $\varphi_i^N(m_1, \ldots, m_k)$ HOLDS, AND SO
BY ABSOLUTENESS DOWN: $\varphi_i^M(m_1, \ldots, m_k)$ HOLDS, I.E.
$\exists x \in M\, \varphi_j^M(x, m_1, \ldots, m_k)$.
ABSOLUTENESS UP: $\varphi_j^N(x, m_1, \ldots, m_k)$ HOLDS
AND SO $(\exists x \in M)\, \varphi_j^N(x, m_1, \ldots, m_k)$

(b) → (a) INDUCTION ON COMPLEXITY: φ_i IS ABSOLUTE FOR M, N
- ATOMIC: BY DEFINITION
- NEGATION AND CONJUNCTION CLEAR
- φ_i IS $(\exists x)\varphi_j(x, y_1, \ldots, y_k)$, LET $m_1, \ldots, m_k \in M$
THEN $\varphi_i^M(m_1, \ldots, m_k) \leftrightarrow (\exists x \in M)\, \varphi_j^M(x, m_1, \ldots, m_k)$
$\leftrightarrow (\exists x \in M)\, \varphi_j^N(x, m_1, \ldots, m_k)$ ← φ_j IS ABSOLUTE
$\leftrightarrow (\exists x \in N)\, \varphi_j^N(x, m_1, \ldots, m_k)$ → ALWAYS ← ASSUMPTION
$\leftrightarrow \varphi_i^M(m_1, \ldots, m_k)$

NOTE THAT (b) MENTIONS φ_j^N ONLY; THIS MAKES CLOSING OFF ARGUMENTS POSSIBLE.

LEMMA [12.15 in JECH]

IF M_0 IS A SET AND $\varphi(x, u_1, \dots, u_k)$ A FORMULA
· THEN THERE IS A SET $M \supseteq M_0$ SUCH THAT
FOR ALL $m_1, \dots, m_k \in M$ IF $(\exists x)\varphi(x, m_1, \dots, m_k)$
THEN $(\exists x \in M)\varphi(x, m_1, \dots, m_k)$

· USING AC WE CAN ACHIEVE $|M| \leq |M_0| \cdot \aleph_0$.

DEFINE A FUNCTION H_φ AS FOLLOWS

$$H_\varphi(u_1, \dots, u_k) = \{x : \varphi(x, u_1, \dots, u_n) \wedge (\forall y)(\varphi(y, u_1, \dots, u_n) \rightarrow \text{RANK } x \leq \text{RANK } y)\}$$

(SO $H_\varphi(u_1, \dots, u_k) = \emptyset$ IF THERE IS NO SUITABLE x)

($H_\varphi(u_1, \dots, u_k)$ IS A SET, A SUBSET OF V_α,
WHERE $\alpha = \min\{\text{RANK } x : \varphi(x, u_1, \dots, u_k)\} + 1$)

SO NOW START WITH M_0 AND RECURSIVELY
SET $M_{i+1} = M_i \cup \bigcup\{H_\varphi(u_1, \dots, u_k) : u_1, \dots, u_k \in M_i\}$
AND, IN THE END $M = \bigcup_{i \in \omega} M_i$.

TO GET A SMALL SET LET $F : \mathcal{P}(M) \setminus \{\emptyset\} \rightarrow M$
BE A CHOICE FUNCTION AND LET

$$h(u_1, \dots, u_k) = \begin{cases} F(H_\varphi(u_1, \dots, u_k)) & \text{IF } H_\varphi(u_1, \dots, u_k) \neq \emptyset \\ u_1 & \text{IF } H_\varphi(u_1, \dots, u_k) = \emptyset \end{cases}$$

START WITH $N_0 = M_0$ AND, RECURSIVELY,
SET $N_{i+1} = N_i \cup h[N_i^k]$
AND IN THE END $N = \bigcup_{i \in \omega} N_i$
NOW SHOW $|N| \leq |M_0| \cdot \aleph_0$.

GENERALIZATIONS:

- FINITELY MANY FORMULAS INSTEAD OF JUST ONE
- M CAN BE OF THE FORM V_α FOR SOME LIMIT α.
  GIVEN M_i LET $\alpha_i = \sup\{\text{RANK } x : x \in \bigcup\{H_\varphi(u_1, \dots, u_k) : u_1, \dots, u_k \in M_i\}\}$
  AND $M_{i+1} = V_{\alpha_i + 1}$.
- IF $M_0 \subseteq L$ THEN M CAN BE OF THE FORM L_α
  FOR SOME LIMIT α, AND THEN WE GET
  IF $(\exists x \in L)\varphi^L(x, u_1, \dots, u_k)$ THEN $(\exists x \in L_\alpha)\varphi^L(x, u_1, \dots, u_k)$

NOW WE CAN FINISH THE INFORMAL PROOF
THAT L IS A MODEL OF ZF.

Let $\varphi(x, z, w_1, \dots, w_k)$ be a formula as in the comprehension axiom.

Let $\varphi_1, \dots, \varphi_n$ be a list of formulas that contains φ and that is closed under taking subformulas.

We had an α and $z, w_1, \dots, w_k \in L_\alpha$ and needed to show $\{x \in z : \varphi^L(x, z, w_1, \dots, w_k)\} \in L$.

Apply the lemma to L_α and the list to find a limit ordinal $\beta > \alpha$ such that for all j we have

$$(\forall m_1, \dots, m_\ell \in L_\beta)\left((\exists x \in L)\, \varphi_j^L(x, m_1, \dots, m_\ell) \to (\exists x \in L_\beta)\, \varphi_j^L(x, m_1, \dots, m_\ell)\right)$$

Then all φ_i, and in particular our φ, are absolute for L_β, L.

And then we know that

$$\{x \in z : \varphi^L(x, z, w_1, \dots, w_k)\} = \{x \in z : \varphi^{L_\beta}(x, z, w_1, \dots, w_k)\} \in L_{\beta+1}.$$

For later use {

Remember the transitive closure [2022-10-24]:

$U^0 x = x$, $U^{n+1} x = U(U^n x)$, .. $\operatorname{trcl} x = \bigcup_{n \in \omega} U^n x$

the smallest transitive set that contains x.

For an infinite cardinal κ define $H(\kappa) = \{x : |\operatorname{trcl}(x)| < \kappa\}$

This is a set: $H(\kappa) \subseteq V_\kappa$

If $x \in H(\kappa)$ then $\operatorname{rank}(x) < \kappa$

Let $S = \{\operatorname{rank}(y) : y \in \operatorname{trcl}(x)\}$ – a set of ordinals

Assume $\alpha \subseteq S$, $\alpha \notin S$, $\beta = \min\{\gamma \in S : \alpha < \gamma\}$ (if possible)

take $y \in \operatorname{trcl}(x)$ with $\operatorname{rank}(y) = \beta$

by transitivity $\{\operatorname{rank}(z) : z \in y\} \subseteq \alpha$

and so $\sup\{\operatorname{rank}(z)+1 : z \in y\} \leq \alpha < \beta = \operatorname{rank}(y)$

contradiction so S is an ordinal

Since $|\operatorname{trcl}(x)| < \kappa$ we have $S < \kappa$

and so $\operatorname{rank}(x) \leq S < \kappa$.

In fact $S = \operatorname{rank}(x) = \{\operatorname{rank}(y) : y \in \operatorname{trcl}(x)\}$.

$H(\aleph_0)$ hereditarily finite sets $H(\aleph_0) = V_\omega$

$H(\aleph_1)$ hereditarily countable sets.

$H(\kappa)$ is a model of $ZF - P$ ($ZFC - P$ if AC).

BACK TO THE DEFINITION OF L.

WE NEED TO SHOW:

IF $\varphi(u_1,\dots,u_m)$ IS A Δ_0-FORMULA THEN THERE IS A COMPOSITION G OF $G_1,\dots,G_{10}$ SUCH THAT FOR ALL $X_1,\dots,X_m$

$$G(X_1,\dots,X_m) = \{\langle u_1,\dots,u_m\rangle : u_1\in X_1\wedge\dots\wedge u_m\in X_m\wedge\varphi(u_1,\dots,u_m)\}.$$

THE PROOF IS BY INDUCTION ON THE COMPLEXITY OF φ.

WE USE ONLY $\neg$, $\wedge$, AND $\exists$

WE SKIP $=$: $x=y$ IS $(\forall u\in x)(u\in y)\wedge(\forall u\in y)(u\in x)$

FIRST $u_i\in u_j$

- $m=2$: $u_1\in u_2$ AND $u_2\in u_1$

  $\{\langle u_1,u_2\rangle : u_1\in u_2\} = G_3(X_1,X_2)$

  $\{\langle u_1,u_2\rangle : u_2\in u_1\} = G_8(G_3(X_2,X_1))$

- $m>2$: VARIOUS SUBCASES
  - $i,j\neq m$.

    $\{\langle u_1,\dots,u_{m-1}\rangle : \dots u_i\in u_j\} = G(X_1,\dots,X_{m-1})$

    $\{\langle u_1,\dots,u_{m-1},u_m\rangle : \dots u_i\in u_j\} = G_2(G(X_1,\dots,X_{m-1}),X_m)$
  - $i,j\neq m-1$

    $\{\langle u_1,\dots,u_{m-2},u_m,u_{m-1}\rangle : \dots u_i\in u_j\} = G(X_1,\dots,X_m)$

    $\{\langle\langle u_1,\dots,u_{m-2}\rangle,u_{m-1},u_m\rangle : \dots u_i\in u_j\} = G_9(G(X_1,\dots,X_m))$
  - $i=m-1,\ j=m$

    $\{\langle u_{m-1},u_m\rangle : u_{m-1}\in u_m\} = G_3(X_{m-1},X_m)$

    $\{\langle\langle u_{m-1},u_m\rangle,\langle u_1,\dots,u_{m-2}\rangle\rangle : \dots u_{m-1}\in u_m\} = G_2(G_3(X_{m-1},X_m),X_1\times\dots\times X_{m-2})$

    $\langle u_{m-1},u_m,\langle u_1,\dots,u_{m-2}\rangle\rangle \longrightarrow \langle\langle u_1,\dots,u_{m-2}\rangle,u_{m-1},u_m\rangle$

    APPLY G_{10} TO GET $\{\langle u_1,\dots,u_{m-2},u_{m-1},u_m\rangle : u_{m-1}\in u_m\}$
  - $i=m,\ j=m-1$

    START WITH $G_3(X_m,X_{m-1})$ ACT AS ABOVE AND AT THE END APPLY G_9.

NEGATION

$$X_1\times\dots\times X_m\setminus G(X_1,\dots,X_m) = G_4(X_1\times\dots\times X_m, G(X_1,\dots,X_m))$$

$$X_1\times X_2 = G_2(X_1,X_2)\quad (X_1\times X_2)\times X_3 = G_2(G_2(X_1,X_2),X_3),\ \dots$$

CONJUNCTION $\quad G_\varphi(X_1,\dots,X_m)\cap G_\psi(X_1,\dots,X_m) = G_5(-,-)$

QUANTIFICATION

φ is $(\exists u_{m+1}\in u_i)\,\psi(u_1,\dots,u_m,u_{m+1})$, i.e.,

$(\exists u_{m+1})(u_{m+1}\in u_i \wedge \psi(u_1,\dots,u_m,u_{m+1}))$

we have $G_a(X_1,\dots,X_{m+1}) = \{\langle u_1,\dots,u_{m+1}\rangle : u_{m+1}\in u_i\}$

$G_b(X_1,\dots,X_{m+1}) = \{\langle u_1,\dots,u_{m+1}\rangle : \psi(u_1,\dots,u_{m+1})\}$

and so $G_5(G_a(-), G_b(-))$ gives

$\{\langle u_1,\dots,u_{m+1}\rangle : u_{m+1}\in u_i \wedge \psi(u_1,\dots,u_{m+1})\}$

Now $\varphi(u_1,\dots,u_m) \leftrightarrow (\exists v\in u_i)(\psi(u_1,\dots,u_m, v))$

$\leftrightarrow (\exists v)(v\in u_i \wedge \psi(\underline{u}, v) \wedge v\in \bigcup X_i)$

$\leftrightarrow \langle u_1,\dots,u_m\rangle \in \mathrm{dom}\left\{\langle \underline{u}, v\rangle : \underline{u}\in X_1\times\dots\times X_m,\ v\in\bigcup X_i,\ v\in u_i \wedge \psi(u_1,\dots,u_m,v)\right\}$

$G_6(X_i)$

$G_7(G_6(X_1,\dots,X_m,\bigcup X_i))$

CONVERSELY IF G IS A COMPOSITION OF THE OPERATIONS $G_1,\dots,G_{10}$ THEN

$$Z = G(X_1,\dots,X_m)$$

CAN BE EXPRESSED BY A Δ_0-FORMULA. (JECH 12.10)

$G_1, G_2, G_3, G_4, G_5, G_6, G_7$ DONE ALREADY

G_8: $Z = G_8(X)$

$(\forall z\in Z)(\exists x\in X)(\exists u\in \mathrm{ran}\,X)(\exists v\in \mathrm{dom}\,X)(x=\langle v,u\rangle \wedge z=\langle u,v\rangle)$

$\wedge (\forall x\in X)(\forall u\in\mathrm{ran}\,X)(\forall v\in \mathrm{dom}\,X)(\exists z\in Z)(x=\langle v,u\rangle \to z=\langle u,v\rangle)$

G_9, G_{10} EXERCISE

BY INDUCTION (i) $u\in G(X_1,\dots)$ is Δ_0

(ii) IF φ IS Δ_0 THEN SO ARE $\forall u\in G(X_1,\dots)\,\varphi$ AND $\exists u\in G(X_1,\dots)\,\varphi$

(iii) $Z = G(X_1,\dots)$ IS Δ_0

(iv) IF φ IS Δ_0 THEN SO IS $\varphi(G(X_1,\dots))$.

(i) SAY $x\in G_3(G(X,\dots), H(X,\dots))$ MEANS

$(\exists u\in G(X,\dots))(\exists v\in H(X,\dots))(u\in v \wedge x=\langle u,v\rangle)$

THEN APPLY INDUCTIVE HYPOTHESIS TO G AND H

(iii) SAY $\varphi(G(X\dots)\cap H(X\dots))$ BECOMES

$\underset{\Delta_0}{\varphi(u)} \wedge \underline{u=(G(X\dots)\cap H(X\dots))}$

$(\forall v\in u)(u\in G(X\dots)\wedge v\in H(X\dots))$

$\wedge\ \forall v\in G(X,\dots)(v\in H(X\dots)\to v\in u)$

(iii) $Z = G(X,-)$: $(\forall u \in Z)(u \in G(X-)) \wedge (\forall u \in G(X-))(u \in Z)$

(iv) IN $\varphi(G(X,-))$ WE HAVE

$u \in G(X-)$ SEE (i), $u = G(X,-)$ SEE (iii)

$\forall u \in G(X-)$ $\exists u \in G(X-)$ SEE (ii)

$G(X,-) \in u$: $(\exists v \in u)(v = G(X,-))$ SEE (iii).

WHY ALL THIS ADO ABOUT Δ_0-FORMULAS?

FOR EVERY FORMULA φ THE RELATIVATION φ^M IS A Δ_0-FORMULA.

- SO FOR EVERY FORMULA φ THERE IS A G SUCH THAT FOR EVERY TRANSITIVE M AND ALL $a_1, \ldots, a_m$

$$\{x \in M : M \models \varphi[x, a_1, \ldots, a_m]\} = \{x \in M : \varphi^M(x, a_1, \ldots, a_m)\}$$
$$= G(M, a_1, \ldots, a_m)$$

- IF G IS A COMPOSITION THEN WE HAVE A Δ_0-FORMULA φ SUCH THAT FOR ALL $M, a_1, \ldots, a_m$

IF $X = G(M, a_1, \ldots, a_m)$ THEN $X = \{x : \varphi(M, x, a_1, \ldots, a_m)\}$

IF M IS TRANSITIVE AND $X \subseteq M$ THEN

$X = \{x \in M : M \models \psi[x, a_1, \ldots, a_m]\}$ $\quad \psi : \exists u \in M \sim \exists u$

THIS GIVES US, FINALLY, A GOOD DESCRIPTION OF $DEF(M)$:

$$DEF(M) = \underbrace{CL(M \cup \{M\})}_{\text{CLOSURE UNDER } G_1, \ldots, G_{10}} \cap \mathcal{P}(M)$$

THIS IS $DEF(M)$ FORMALIZED WITHIN ZF.

SO NOW $\langle L_\alpha : \alpha \in On \rangle$ IS WELL-DEFINED

WHAT IS LEFT TO DO?

WE WANT TO SHOW

- $\langle L_\alpha : \alpha \in On \rangle$ IS ABSOLUTE FOR TRANSITIVE MODELS OF ZF − P
- SO THAT $(\exists \alpha)(x \in L_\alpha)$ WILL BE ABSOLUTE TOO
- L CAN BE WELL-ORDERED, SO $L \models AC$
- L SATISFIES GCH
- L HAS A SOUSLIN TREE (IF THERE IS TIME)

WE HAVE Δ_0-FORMULAS

WE HAVE Σ_1- AND Π_1-FORMULAS

Σ_1 : $(\exists x)\varphi$ WITH φ A Δ_0-FORMULA

Π_1 : $(\forall x)\varphi$ WITH φ A Δ_0-FORMULA

A PROPERTY IS Σ_1 OR Π_1 IF IT CAN BE EXPRESSED BY A Σ_1- OR Π_1-FORMULA

IT IS Δ_1 IF IT CAN BE EXPRESSED BY A Σ_1- AND A Π_1-FORMULA.

- Σ_1-FORMULAS ARE UPWARD ABSOLUTE
- Π_1-FORMULAS ARE DOWNWARD ABSOLUTE

SO Δ_1-PROPERTIES ARE ABSOLUTE.

WE WANT TO SHOW $\alpha \mapsto L_\alpha$ IS Δ_1

① $Y = \mathrm{DEF}(X)$ IS Σ_1

$$\exists W \big(W \text{ IS A FUNCTION} \wedge \mathrm{DOM}\, W = \omega \wedge W(0) = X \wedge \wedge (\forall n \in \omega)\big(W(n+1) = W(n) \cup \{G_i(x,y) : x \in W(n) \wedge y \in W(n) \wedge i \in 11 \setminus 1\}\big) \wedge \wedge Y = \bigcup \mathrm{RAN}\, W \big)$$

EVERYTHING IN THE PARENTHESES IS Δ_0.

② RECURSION APPLIED TO Σ_1-FUNCTIONS RESULTS IN A Σ_1-FUNCTION

$$x = L_\alpha \leftrightarrow \exists f \big(f \text{ IS A FUNCTION} \overset{\Delta_0}{\wedge} \mathrm{DOM}\, f \overset{\Delta_0}{=} \alpha \wedge f(0) \overset{\Delta_0}{=} \emptyset \wedge (\forall \beta \in \alpha)(f(\beta+1) \overset{\Sigma_1}{=} \mathrm{DEF}(f[\beta] \cup \{f[\beta]\})) \wedge (\forall \beta \in \alpha)(\beta \overset{\Delta_0}{=} \bigcup \beta \to f(\beta) \overset{\Delta_0}{=} \bigcup \mathrm{RAN}\, f \restriction \beta) \wedge (\alpha \overset{\Delta_0}{=} \bigcup \alpha \to x \overset{\Delta_0}{=} \bigcup \mathrm{RAN}\, f) \big)$$

③ BUT $x = L_\alpha$ IS EQUIVALENT TO

$$\alpha \in \mathrm{On} \wedge (\forall z)(z = L_\alpha \to x = z)$$

THIS IS Π_1 BECAUSE $z = L_\alpha \to x = z$ IS Π_1

BY TRANSLATION $\neg(z = L_\alpha) \vee x = z$

$\neg(z = L_\alpha) = \neg \exists f\, \varphi(f) = \forall f (\neg \varphi(f))$ IS Π_1

SO $x = L_\alpha$ IS ALSO Π_1-EXPRESSIBLE

④ DONE: $\alpha \mapsto L_\alpha$ IS Δ_1 AND SO ABSOLUTE FOR TRANSITIVE MODELS

So $\alpha \mapsto L_\alpha$ is absolute for transitive models

Hence so is

x is constructible

Let M be such a model with $On \subseteq M$.

For $x \in M$ we have

$(x \text{ is constructible})^M$ iff $(\exists \alpha \in M)(x \in L_\alpha^M)$

iff $(\exists \alpha)(x \in L_\alpha)$

iff x is constructible

We get

Theorem [Gödel]

- L satisfies the Axiom of Constructibility

  If $x \in L$ then $(x \text{ is constructible})^L$

  iff x is constructible

  So L satisfies "every set is constructible"

  We shorten that to $V = L$.
- L is the smallest inner model of ZF

  Inner model: it contains On

  If M is an inner model then $L^M = L$

  and so $L \subseteq M$.

Next time AC and GCH in L.

On 2022-12-07: 149 years of uncountability.