WHAT WE KNOW ABOUT L THUS FAR

DEFINITION

$L_0 = \emptyset$, $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$

$= \mathrm{cl}(L_\alpha \cup \{L_\alpha\}) \cap \mathcal{P}(L_\alpha)$

WHERE cl IS THE CLOSURE UNDER THE GÖDEL-OPERATIONS

$L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ IF α IS A LIMIT.

THE FUNCTION $\alpha \mapsto L_\alpha$ IS Δ_1 AND HENCE ABSOLUTE FOR TRANSITIVE MODELS OF ZF-P.

HENCE "x IS CONSTRUCTIBLE" IS ABSOLUTE TOO.

IN PARTICULAR FOR $x \in L$:

x IS CONSTRUCTIBLE IFF (x IS CONSTRUCTIBLE)^L

AND SO L SATISFIES $V = L$.

THERE IS MORE:

LET φ BE THE CONJUNCTION OF ALL THE FINITELY MANY AXIOMS OF ZF-P THAT WE USED TO PROVE THE ABSOLUTENESS OF ORDINALS, RANKS, $\alpha \mapsto L_\alpha$, ... FOR TRANSITIVE MODELS.

THEN: IF M IS A TRANSITIVE PROPER CLASS AND φ^M HOLDS THEN $L \subseteq M$.

NOW WE LOOK AT TRANSITIVE SETS

IF M IS TRANSITIVE THEN $o(M) = M \cap On$ IS AN ORDINAL, THE FIRST ORDINAL NOT IN M.

ADD MORE AXIOMS TO φ TO GET ψ SUCH THAT ψ IMPLIES THERE IS NO LARGEST ORDINAL

IF M IS TRANSITIVE AND ψ^M HOLDS THEN $o(M)$ IS A LIMIT ORDINAL AND $L(o(M)) = \bigcup_{\alpha\in M} L_\alpha$

AND SO

$$L^M = \{x \in M : ((\exists\alpha)(x \in L_\alpha))^M\} = \bigcup_{\alpha\in M} L_\alpha.$$

WE GET

$$L^M = L(o(M)) \subseteq M.$$

Now let $\chi = \psi + (V = L)$

- If M is a transitive proper class and χ^M then $M = L$
- If M is a transitive set and χ^M then $M = L(o(M))$, and $o(M)$ is a limit

This we will use in the proof of GCH.

The Axiom of Choice in L

There is a sequence $\langle <_\alpha : \alpha \in On \rangle$ such that for all α and β

- $<_\alpha$ is a well-order on L_α
- If $\alpha < \beta$ then
  - $x <_\alpha y$ implies $x <_\beta y$ $\quad (x, y \in L_\alpha)$
  - $x \in L_\alpha$ and $y \in L_\beta \setminus L_\alpha$ implies $x <_\beta y$

(This implies $x \in y \in L_\alpha \Rightarrow x <_\alpha y$)

If α is a limit then

$$<_\alpha = \bigcup_{\beta < \alpha} <_\beta$$

(or $<_\alpha = \{ \langle x, y \rangle \in L_\alpha^2 : (\exists \beta \in \alpha)(x <_\beta y) \}$)

From α to $\alpha + 1$

We have $<_\alpha$ and we know

$$L_{\alpha+1} = \mathcal{P}(L_\alpha) \cap cl(L_\alpha \cup \{L_\alpha\}) = \mathcal{P}(L_\alpha) \cap \bigcup_{m \in \omega} W_m^\alpha$$

$W_0^\alpha = L_\alpha \cup \{L_\alpha\}$

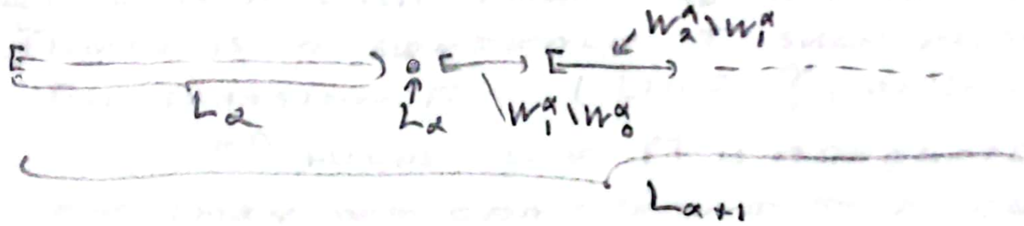
$W_{m+1}^\alpha = \{ G_i(x, y) : x, y \in W_m^\alpha \wedge i \in 11 \setminus 1 \} \cup W_m^\alpha$.

Build $<_{\alpha+1}^m$ on W_m^α, by recursion on m:

- $<_{\alpha+1}^0 = <_\alpha \cup \{ \langle x, L_\alpha \rangle : x \in L_\alpha \}$.
- $x <_{\alpha+1}^{m+1} y$ iff
  - $x <_{\alpha+1}^m y$ $\quad (x, y \in W_m^\alpha)$ or
  - $x \in W_m^\alpha \wedge y \in W_{m+1}^\alpha \setminus W_m^\alpha$ or
  - $x, y \in W_{m+1}^\alpha \setminus W_m^\alpha$ and
    - $\min\{ i : (\exists u, v \in W_m^\alpha)(x = G_i(u, v)) \} < \min\{ j : (\exists u, v \in W_m^\alpha)(y = G_j(u, v)) \}$. or
    - $\min\{ u : (\exists v \in W_m^\alpha)(x = G_i(u, v)) \} <_m^\alpha \min\{ u : (\exists v \in W_m^\alpha)(y = G_i(u, v)) \}$ or
    - $\min\{ v : x = G_i(u, v) \} <_m^\alpha \min\{ v : y = G_i(u, v) \}$.

IN THE END

$$<_{\alpha+1} \;=\; \Big(\bigcup_{n\in\omega} <^n_{\alpha+1}\Big) \cap \big(\mathcal{P}(L_\alpha)\times\mathcal{P}(L_\alpha)\big)$$



AND IN THE VERY END

$$<_L \;=\; \bigcup_{\alpha\in On} <_\alpha$$

THIS RELATION WELL-ORDERS THE WHOLE CLASS L.

THE FUNCTION $\alpha \mapsto <_\alpha$ IS Σ_1

THIS IS LIKE PROVING $\alpha \mapsto L_\alpha$ IS Σ_1:

$(\exists f)(\exists W)\,($ W AS IN $\alpha \mapsto L_\alpha$ BUILDS $cl(L_\alpha \cup \{L_\alpha\})$ $\wedge$

$\wedge$ f IS A FUNCTION $\wedge$ $\operatorname{dom} f = \omega$ $\wedge$

$f(0) = <_\alpha \cup \{\langle x, L_\alpha\rangle : x \in L_\alpha\}$ $\wedge$

$\forall x, y \in W(n+1)\,(\langle x,y\rangle \in f(n+1) \leftrightarrow$

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COMPARISON OF THE MINIMA:

$(\exists u, v)(x = G_i(u,v) \wedge (\forall s,t)(y = G_j(s,t) \rightarrow i < j))$

$(\exists v)(x = G_i(u,v) \wedge \forall s)((\exists t)(y = G_i(s,t) \rightarrow u <^n_{\alpha+1} s)))$

ETC ALL QUANTIFIERS BOUNDED BY $W(n)$.

SO $\langle <_\alpha : \alpha \in On\rangle$ IS ABSOLUTE FOR ALL M SUCH THAT χ^M HOLDS.

ALTERNATIVE FROM JECH (13.12) AND (13.13)

M IS ADEQUATE IF M IS TRANSITIVE AND

- M IS CLOSED UNDER $G_1, \ldots, G_{10}$
- IF $X \in M$ THEN $\{G_i(x,y) : x, y \in X,\ i \in \{1,2\}\} \in M$
- IF $\alpha \in M$ THEN $\langle L_\beta : \beta < \alpha\rangle \in M$

THE Δ_1-FUNCTIONS $\alpha \mapsto L_\alpha$ AND $\alpha \mapsto <_\alpha$
ARE ABSOLUTE FOR ADEQUATE M.
IF δ IS A LIMIT THEN L_δ IS ADEQUATE
THERE IS A SENTENCE σ SUCH THAT
FOR TRANSITIVE M: (LIKE χ, WHICH IS FROM KUNEN)
σ^M HOLDS IFF M IS ADEQUATE
AND THEN WE CAN IMPROVE σ TO GET
σ^M HOLDS IFF $M = L_\delta$ FOR SOME LIMIT δ

THIS LEADS TO GÖDEL'S CONDENSATION LEMMA.
IF δ IS A LIMIT ORDINAL AND IF $M \prec (L_\delta, \in)$
THEN THERE ARE A MAP π AND A $\beta \in \delta$
SUCH THAT $\pi : (M, \in) \to (L_\beta, \in)$ IS AN ISOMORPHISM.

① WHAT DOES $\prec$ MEAN?
IN THIS CASE: FOR EVERY FORMULA φ
AND ELEMENTS $m_1, \dots, m_n$ OF M
$$\varphi^M(m_1, \dots, m_n) \Longleftrightarrow \varphi^{L_\delta}(m_1, \dots, m_n)$$
MODEL THEORY:
$A \prec B$ IF A IS AN ELEMENTARY
SUBSTRUCTURE OF B.

② FOR NOW: M IS WELL-FOUNDED (BY FOUNDATION)
AND EXTENSIONAL:
IF $x, y \in M$ THEN $x = y$ IFF $x \cap M = y \cap M$.
IF $x \neq y$ THEN THERE IS $z \in L_\delta$ SUCH
THAT $z \in x \Leftrightarrow z \notin y$
SO $(\exists z \in L_\delta)\, \varphi^{L_\delta}(x, y, z)$ AND BY $\prec$
WE GET $(\exists z \in M)\, \varphi^{L_\delta}(x, y, z)$
AND $(z \in x \Leftrightarrow z \notin y)^M$ IS JUST $z \in x \Leftrightarrow z \notin y$.

③ π IS THE TRANSITIVE COLLAPSE OF M
BY RECURSION $\pi(x) = \{\pi(y) : y \in M \cap x\}$.
THE RANGE N OF π IS TRANSITIVE
AND $\pi : (M, \in) \to (N, \in)$ IS AN ISOMORPHISM.
NOW: WE KNOW χ^{L_δ} (OR σ^{L_δ}) HOLDS
BY ABSOLUTENESS WE GET χ^M (OR σ^M)
BY ISOMORPHISM WE GET χ^N (OR σ^M)
IN EITHER CASE $N = L_\gamma$ FOR SOME LIMIT γ.

GCH IN L

- WE PROVED, INFORMALLY, THAT $|L_\alpha| = |\alpha|$ FOR INFINITE α.
- BECAUSE AC HOLDS IN L THAT PROOF GOES THROUGH, PROVIDED WE LOOK AT THE CLOSURE OF $L_\alpha \cup \{L_\alpha\}$ UNDER THE GÖDEL OPERATIONS [LEMMA 13.14 IN JECH]
  IN GENERAL $|cl(M)| \leq |M| \cdot \aleph_0$.
- IN PARTICULAR IF κ IS AN INFINITE CARDINAL THEN $|L_{\kappa^+}| = \kappa^+$
- WE SHALL BE DONE IF WE SHOW THAT, IN L, $\mathcal{P}(\kappa) \subseteq L_{\kappa^+}$.
- LET $X \subseteq \kappa$ BE CONSTRUCTIBLE
  LET $\delta > \kappa$ BE A LIMIT ORDINAL SUCH THAT $X \in L_\delta$.
  LET $M \prec L_\delta$ BE SUCH THAT $L_\kappa \cup \{X\} \subseteq M$ AND $|M| = \kappa$. [HOW? IN A MOMENT.]
  LET $\pi : M \to L_\gamma$ BE AN ISOMORPHISM
  THEN $\gamma \leq \delta$ AND $\gamma < \kappa^+$ BECAUSE $|\gamma| = |L_\gamma| = |M| = \kappa$.
- NOW, BECAUSE $\kappa \subseteq L_\kappa \subseteq M$
  WE GET
  $$\begin{aligned} \pi(X) &= \{\pi(\alpha) : \alpha \in X \cap M\} \\ &= \{\pi(\alpha) : \alpha \in X\} \\ &= \{\alpha : \alpha \in X\} \\ &= X \end{aligned}$$
  AND SO $X \in L_\gamma \subseteq L_{\kappa^+}$.

SO, HOW TO GET $M \prec L_\delta$, OR $M \prec N$ IN GENERAL?

LAST WEEK WE HAD, FOR A SUBFORMULA-CLOSED FINITE SET $\varphi_1, \ldots, \varphi_m$:

- $\varphi_1, \ldots, \varphi_m$ ARE ABSOLUTE FOR M, N IFF
- IF $\varphi_i(y_1, \ldots, y_k)$ IS $(\exists x)\varphi_j(x, y_1, \ldots, y_k)$ THEN FOR ALL $m_1, \ldots, m_k \in M$: $(\exists x \in N)\varphi_j^N(x, m_1, \ldots, m_k) \to (\exists x \in M)\varphi_j^N(x, m_1, \ldots, m_k)$

THE SAME HOLDS FOR $\prec$:

WE HAVE $M \prec N$ IFF FOR ALL $\varphi(x, y_1, \dots, y_k)$
FOR ALL $m_1, \dots, m_k \in M$
$$(\exists x \in N)\, \varphi^N(x, m_1, \dots, m_k) \to (\exists x \in M)\, \varphi^N(x, m_1, \dots, m_k)$$

SO, HOW DO WE DEAL WITH ALL FORMULAS?
AGAIN, BACK TO LAST WEEK.
THE FORMULA $\varphi^N(x, y_1, \dots, y_k)$ IS Δ_0
WE CAN WRITE
$$\{x \in N : \varphi^N(x, m_1, \dots, m_k)\}$$
AS $G(N, \{m_1\}, \dots, \{m_k\})$ WITH G A COMPOSITION OF GÖDEL OPERATIONS

ENUMERATE THE COMPOSITIONS AS $\langle H_n : n \in \omega \rangle$
HOW? VIA AN ENUMERATION OF $\{1,2,3,4,5,6,7,8,9,10\}^{<\omega}$ AND A MAP THAT TURNS EACH SEQUENCE INTO A COMPOSITION.
WELL-ORDER N USING $\triangleleft$ SAY.
DEFINE $F_n : N^{k_n} \to N$ BY
$$F_n(m_1, \dots, m_{k_n}) = \begin{cases} \min H_n(N, \{m_1\}, \dots, \{m_{k_n}\}) & \text{IF THIS SET IS NONEMPTY} \\ \min N & \text{OTHERWISE} \end{cases}$$

GIVEN $M_0 \subseteq N$ LET M BE THE CLOSURE OF M_0 UNDER THE MAPS F_n.
THEN $|M| \leq |M_0| \cdot \aleph_0$ AND $M \prec N$.
BECAUSE WE HAVE MADE SURE THAT THE CONDITION ABOVE HOLDS.

PROPOSITION IF $V = L$ THEN $H(\kappa) = L_\kappa$ WHENEVER κ IS REGULAR

- $L_\omega = H(\aleph_0) = V_\omega$ SO EASY FOR $\kappa = \aleph_0$
- IF $x \in L_\kappa$ THEN $x \in L_\alpha$ FOR SOME $\alpha < \kappa$ AND $\mathrm{TRCL}(x) \subseteq L_\alpha$ AND $|\mathrm{TRCL}(x)| \leq |L_\alpha| \leq \alpha < \kappa$.

So $L_\kappa \subseteq H(\kappa)$

IF $L_\kappa \neq H(\kappa)$ THEN WE CAN TAKE

$A \in H(\kappa) \setminus L_\kappa$ SUCH THAT $A \cap (H(\kappa) \setminus L_\kappa) = \emptyset$

(BY FOUNDATION).

BUT THEN $A \subseteq H(\kappa) \cap L_\kappa = L_\kappa$

BY REGULARITY OF κ WE GET $\alpha < \kappa$

SUCH THAT $A \subseteq L_\alpha$

USE THE PROOF OF GCH WE GET

$A \in L_{\alpha^+} \subseteq L_\kappa$, CONTRADICTION.

ALSO: $\{\alpha \in \omega_1 : L_\alpha \prec L_{\omega_1}\}$ IS CUB IN ω_1

- USE CLOSURE UNDER FUNCTIONS
- AND IF $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots$

  AND $L_{\alpha_n} \prec L_{\omega_1}$ FOR ALL n

  THEN $L_{\sup_n \alpha_n} \prec L_{\omega_1}$.

So (!) IF $L_\alpha \prec L_{\omega_1}$ THEN L_α SATISFIES

$ZF - P + V = L$ BECAUSE $L_{\omega_1} = H(\aleph_1)$ DOES.