Set Theory
2010/2011; 2nd Semester
K. P. Hart & B. Löwe

Written Exam: 22 June 2011, 13-16, room

Name:

University:

Student ID:

General comments.

(1) **The time for this exam is 3 hours (180 minutes).**
(2) **There are 104 points in the exam: 52 points are sufficient for passing.**
(3) **Please mark the answers to the questions in Exercise I on this sheet by crosses.**
(4) **Make sure that you have your name, university and student ID on each of the sheets you are handing in.**
(5) **If you have any questions, please indicate this silently and someone will come to you. Answers to questions that are relevant for anyone will be announced publicly.**
(6) **No talking during the exam.**
(7) **Cell phones must be switched off and stowed.**

Exercise I (24 points).
Find the correct answer (1 point each):

(1) One of the following four theories cannot have finite models (any model is non-empty, so $\exists x(x = x)$ is always true; also Empty Set refers to the axiom $\exists x \forall y(y \notin x)$). Which one?
- ☐ **A:** Extensionality + Separation.
- ☐ **B:** Extensionality + AC.
- ☐ **C:** Pairing + AC.
- ☐ **D:** Extensionality + Empty Set + Pairing.

(2) Who gave the first axiomatic treatment of set theory, leading to a predecessor of the current standard axiomatic system ZFC?
- ☐ **A:** Georg Cantor.
- ☐ **B:** Ernst Zermelo.
- ☐ **C:** Abraham Fraenkel.
- ☐ **D:** Saharon Shelah.

(3) One of the following ordinal equalities is false. Which one?
☐ **A:** $2 \cdot \omega = 3 \cdot \omega$.
☐ **B:** $3 + \omega + \omega^2 = \omega + 3 + \omega^2$.
☐ **C:** $\omega^2 + 3 = 3 + \omega^2$.
☐ **D:** $12 \cdot (5 + \omega) = 60 \cdot \omega$.

(4) In the usual formalization of natural numbers and ordered pairs (i.e., $n = \{0, ..., n-1\}$ and $(x, y) := \{\{x\}, \{x, y\}\}$), one of the following statements is true. Which one?
☐ **A:** $17 \in 4$.
☐ **B:** $2 \in (0, 1)$.
☐ **C:** $(0, 1) = 2$.
☐ **D:** $4 \in (4, 17)$.

(5) Which of the following inequalities is true in *ordinal arithmetic*?
☐ **A:** $2^\omega < 3^\omega$.
☐ **B:** $\omega^2 < \omega^3$.
☐ **C:** $2 + \omega < 3 + \omega$.
☐ **D:** $2 \cdot \omega < 3 \cdot \omega$.

(6) Which of the following is provable in ZF?
☐ **A:** If κ is an infinite *cardinal number* then $\kappa \cdot \kappa = \kappa$.
☐ **B:** If κ is an infinite *cardinality* then $\kappa \cdot \kappa = \kappa$.
☐ **C:** If κ is an infinite *cardinality* then $\aleph_0 \leq \kappa$.
☐ **D:** If κ is an infinite *cardinal number* then so is 2^κ.

(7) Consider the statement "for all limit ordinals λ, $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$". Which of the following statements is true?
☐ **A:** The statement is provable in ZF.
☐ **B:** The statement is provable in ZFC, but not in ZF.
☐ **C:** The statement is refutable.
☐ **D:** The statement has large cardinal strength.

(8) Suppose that λ is a limit cardinal. Which of the following statements is provable in ZFC?
☐ **A:** The cardinal λ is singular.
☐ **B:** The cardinal λ is regular.
☐ **C:** $\lambda = \aleph_\lambda$.
☐ **D:** None of the above.

(9) Assume $\aleph_\omega$ is a strong limit, which of the following is not provable in ZFC?
☐ **A:** $2^{\aleph_\omega} = \aleph_\omega^{\aleph_0}$
☐ **B:** $2^{\aleph_\omega} = \max \mathrm{pcf}\{\aleph_n : n \in \omega\}$
☐ **C:** $2^{\aleph_\omega} < \aleph_{\omega_4}$
☐ **D:** $2^{\aleph_\omega} \leq \aleph_{\omega_3}$

(10) The Continuum Hypothesis CH states that $2^{\aleph_0} = \aleph_1$. One of the following statements is equivalent to CH (in ZFC). Which one?
☐ **A:** Every uncountable set of real numbers contains a perfect set.
☐ **B:** Every uncountable set of real numbers contains a set of cardinality $2^{\aleph_0}$.
☐ **C:** Every perfect set is uncountable.
☐ **D:** Every uncountable set has cardinality $2^{\aleph_0}$.

(11) Which of the following statements are provable in ZF for all *cardinalities*?
☐ **A:** $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.
☐ **B:** $\kappa \leq \lambda$ implies $\kappa^\mu \leq \lambda^\mu$.
☐ **C:** $\kappa \leq \lambda$ implies $\mu^\kappa \leq \mu^\lambda$.
☐ **D:** $\kappa < \lambda$ implies $\mu^\kappa < \mu^\lambda$.

(12) On the basis of ZF, there are many equivalents of the Axiom of Choice. One of the following theorems of ZFC is **not** equivalent to the Axiom of Choice. Which one?
☐ **A:** Zorn's Lemma.
☐ **B:** Zermelo's Wellordering Theorem.
☐ **C:** Every vector space has a basis.
☐ **D:** There is a non-Lebesgue measurable set.

(13) Let A and B be infinite sets. Which of the following statements is, in ZF, equivalent to $|A| \leq |B|$:
☐ **A:** $|\mathcal{P}(A)| \leq |\mathcal{P}(B)|$.
☐ **B:** There is a surjection $f : B \to A$.
☐ **C:** $|A \times A| \leq |B|$.
☐ **D:** None of the above.

(14) Which of the following is *not* provable in ZF?
☐ **A:** There is an injection $f : \omega_1 \to \mathbb{R}$.
☐ **B:** There is a surjection $f : \mathbb{R} \to \omega_1$.
☐ **C:** There is a bijection $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$.
☐ **D:** There is a bijection $f : \mathbb{N} \to \mathbb{A}$, where $\mathbb{A}$ is the set of algebraic numbers.

(15) The fact that ω_1 is regular is a theorem of ZFC, but cannot be proved in ZF alone. In class, we introduced $\mathsf{AC}_X(Y)$ for "for every function $f : X \to \wp(Y)\backslash\{\varnothing\}$ there is a function $g : X \to Y$ such that $\forall x(g(x) \in f(x))$" and showed that AC is equivalent to $\forall X \forall Y \mathsf{AC}_X(Y)$. Only of the following fragments of AC can prove that ω_1 is regular. Which one?
☐ **A:** $\mathsf{AC}_{\omega_1}(\omega)$.
☐ **B:** $\mathsf{AC}_{\omega}(\omega_1)$.
☐ **C:** $\mathsf{AC}_{\omega}(\mathcal{P}(\omega))$.
☐ **D:** $\mathsf{AC}_2(\wp(\omega_1))$.

(16) Which of the following partition relations is not provable in ZFC?
☐ **A:** $2^{\aleph_1} \to (\aleph_2)^2_2$?
☐ **B:** $(2^{\aleph_1})^+ \to (\aleph_2)^2_{\aleph_1}$?
☐ **C:** $\aleph_0 \to (\aleph_0)^{25}_{300}$?
☐ **D:** $\aleph_2 \to (\aleph_0, \aleph_2)^2$?

(17) Only one of the following statements of cardinal arithmetic is provable in ZFC. Which one? (κ, λ and μ are assumed to be infinite cardinals)
☐ **A:** $2^{\aleph_0} = \aleph_2$.
☐ **B:** $\kappa^{\mathrm{cf}\,\kappa} = 2^\kappa$.
☐ **C:** If $\mu < \kappa$ and $\mu^\lambda \geq \kappa$ then $\kappa^\lambda = \mu^\lambda$.
☐ **D:** $2^{<\kappa} = \kappa$.

(18) Let M be a countable elementary substructure of $\mathbf{H}(\aleph_3)$. Which of the following statements is true?
☐ **A:** $\omega \in M$.
☐ **B:** $\omega_1 \in M$.
☐ **C:** $\omega_2 \in M$.

☐ **D:** $\omega_1 \subseteq M$.

(19) Which of the following statements is true?

☐ **A:** You cannot prove the existence of strong limit cardinals in ZFC.
☐ **B:** You can prove the existence of strong limit cardinals of cofinality ω_2 in ZFC.
☐ **C:** If κ is a strong limit, then $\kappa = 2^\kappa$.
☐ **D:** It is consistent that $\aleph_2$ is a strong limit cardinal.

(20) As in the lecture, we say that for theories S and T, S *is stronger than* T if S proves the consistency of T. Which of the following theories is the strongest?

☐ **A:** There are two inaccessible cardinals.
☐ **B:** There are infinitely many inaccessible cardinals.
☐ **C:** There are two inaccessible cardinals $\kappa < \lambda$ and κ is itself a limit of inaccessible cardinals.
☐ **D:** There is a proper class of inaccessible cardinals.

(21) Which of the following is provable in ZFC?

☐ **A:** If the GCH holds below $\aleph_\omega$ then $2^{\aleph_\omega} = \aleph_{\omega+1}$.
☐ **B:** If the GCH holds below $\aleph_{\omega_1}$ then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.
☐ **C:** If the GCH holds below $\aleph_{25}$ then $2^{\aleph_{25}} = \aleph_{26}$.
☐ **D:** If $\aleph_{\omega_1}$ is a strong limit then $2^{\aleph_{\omega_1}} \leq \aleph_{\omega_2}$.

(22) One of the following statements is true. Which one?

☐ **A:** Every inaccessible cardinal is weakly compact.
☐ **B:** Every weakly compact cardinal is inaccessible.
☐ **C:** Every Mahlo cardinal is measurable.
☐ **D:** Every inaccessible cardinal is Mahlo.

(23) Let κ be a regular cardinals. One of the following sets forms a filter on κ. Which one?

☐ **A:** $\{C\,;\, C \text{ is club in } \kappa\}$.
☐ **B:** $\{C\,;\, C \text{ contains a set that is club in } \kappa\}$.
☐ **C:** $\{C\,;\, C \text{ is stationary in } \kappa\}$.
☐ **D:** $\{C\,;\, C \text{ is not club in } \kappa\}$.

(24) One of the following statements is false. Which one?

☐ **A:** If κ has the tree property, then κ is weakly compact.
☐ **B:** If κ is weakly compact, then κ has the tree property.
☐ **C:** If κ is weakly compact, then every $\mathcal{L}_{\kappa,\omega}$-language as the weak compactness property.
☐ **D:** If κ is weakly compact, then it is the κth inaccessible cardinal.

Exercise II (15 points).

Work without using the Axiom of Choice. Prove that $\aleph_2$ cannot be a countable union of countable sets.

[**Hint.** If $S \subseteq \aleph_2$ is countable, then the order type of $(S, <)$ is a countable ordinal.]

Exercise III (15 points).

Prove the first non-trivial version of Ramsey's theorem:

$$\aleph_0 \to (\aleph_0)^2_2.$$

Exercise IV (20 points).

Work without the Axiom of Choice. For a set X, let $h(X)$ be the first ordinal α for which there is *no* injection $f : \alpha \to X$.

(1) Prove that $h(X)$ is well-defined (5 points).
(2) Prove: X is well-orderable if and only if there is an injection from X into $h(X)$ (5 points).
(3) Prove: if there is a bijection $b : (X \cup h(X))^2 \to X \cup h(X)$ then there is an injection $f : X \to h(X)$ (5 points).
[**Hint.** Show that for every $x \in X$ there is an $\alpha_x < h(X)$ such that $b(x, \alpha_x) \notin X$.]
(4) Prove that AC is equivalent to the statement that $\kappa \cdot \kappa = \kappa$ for all infinite *cardinalities* (5 points).

Exercise V (30 points).

Recall that a class function $S : \mathrm{Ord} \to \mathrm{Ord}$ was called a *normal sequence* if it is monotone (for $\alpha < \beta$, we have $S(\alpha) < S(\beta)$) and continuous (for limit λ, we have $S(\lambda) = \bigcup\{S(\alpha)\,;\,\alpha < \lambda\}$).

(1) Prove that every normal sequence has arbitrarily large fixed points (i.e., for every α, there is a $\xi > \alpha$ such that $S(\xi) = \xi$ (5 points).
(2) Consider the sequence recursively defined by $\beth_0 := \omega$, $\beth_{\alpha+1} := 2^{\beth_\alpha}$, and $\beth_\lambda := \bigcup\{\beth_\alpha\,;\,\alpha < \lambda\}$. Prove that there is a so-called *beth fixed point*, i.e., a κ such that $\beth_\kappa = \kappa$ (5 points).
(3) Prove that the least beth fixed point has cofinality ω (5 points).
(4) Suppose that there are unboundedly many inaccessible cardinals. Consider the class function $\iota : \mathrm{Ord} \to \mathrm{Ord}$ such that $\iota(\alpha)$ is the αth inaccessible cardinal. Show that ι is not normal (5 points).
(5) Suppose now that there is an ι-fixed point, i.e., a κ such that $\iota(\kappa) = \kappa$. Prove that the least such κ is not a Mahlo cardinal (10 points).