TU Delft — mastermath — Universiteit van Amsterdam

# Set Theory

**2010/2011; 2nd Semester**

**K. P. Hart & B. Löwe**

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**Name:**

**University:**

**Student ID:**

**General comments.**

(1) **The time for this exam is 3 hours (180 minutes).**
(2) **There are 104 points in the exam: 52 points are sufficient for passing.**
(3) **Please mark the answers to the questions in Exercise I on this sheet by crosses.**
(4) **Make sure that you have your name, university and student ID on each of the sheets you are handing in.**
(5) **If you have any questions, please indicate this silently and someone will come to you. Answers to questions that are relevant for anyone will be announced publicly.**
(6) **No talking during the exam.**
(7) **Cell phones must be switched off and stowed.**

**Exercise I** (25 points).

Find the correct answer (1 point each):

(1) Consider the structure $\mathbf{V}_\omega$, with $\in$ and $=$ as usual. Which of the following axioms of ZFC does not hold in this structure?
☐ A Axiom of Replacement,
☐ B Axiom of Choice,
☐ C Axiom of Infinity,
☐ D Axiom of Pairing.

(2) Consider the statement "for all ordinals $\alpha$ one has $\operatorname{cf}\aleph_\alpha = \operatorname{cf}\alpha$". Which of the following is true?
☐ A The statement is provable in ZF.
☐ B The statement is provable in ZFC but not in ZF.
☐ C The statement is refutable.
☐ D The statement has large cardinal strength.

(3) One of the following ordinal inequalities is correct. Which one?
☐ A $\omega \cdot 2 = \omega \cdot 3$.
☐ B $\omega^2 + \omega + 3 = \omega^2 + 3 + \omega$.
☐ C $\omega + \omega^2 + \omega^3 = \omega^2 + \omega + \omega^3$.
☐ D $12 \cdot (\omega + 5) = \omega \cdot 60$.

(4) We adopt the usual definitions of ordered pair as $(a, b) = \{\{a\}, \{a, b\}\}$ and ordinal numbers as their sets of predecessors. Just one of the following statements is not true, which one?
☐ A $1 \in (0, 1)$.
☐ B $2 \in (0, 1)$.
☐ C $0 \in (0, 1)$.
☐ D $(0, 1) \subseteq 3$.

(5) Which of the following inequalities is false in *ordinal arithmetic*?
☐ A $2^\omega < 3^\omega$.
☐ B $\omega^2 < \omega^3$.
☐ C $\omega + 2 < \omega + 3$.
☐ D $\omega \cdot 2 < \omega \cdot 3$.

(6) Define the following structure: let the universe be the set $\mathbb{N}$ of natural numbers; interpret $\in$ by $<$ and let $=$ be ordinary equality. Which of the following axioms holds in this structure:
☐ A Pairing
☐ B Separation
☐ C Union
☐ D Infinity

(7) Which of the following is provable in $\mathsf{ZF}$?
☐ A If $\kappa$ is an infinite *cardinality* then $\aleph_0 \leq \kappa$.
☐ B If $\kappa$ is an infinite *cardinal number* then $\aleph_0 \leq \kappa$.
☐ C If $\kappa$ is an infinite *cardinality* then $\kappa \cdot \kappa = \kappa$.
☐ D If $\kappa$ is an infinite *cardinal number* then so is $2^\kappa$.

(8) One of the following cardinal numbers is *not* regular, which one?
☐ A 0.
☐ B 1.
☐ C 2.
☐ D $\aleph_0$.

(9) Assume $\aleph_{\omega_1}$ is a strong limit. Which of the following is (not) provable in $\mathsf{ZFC}$
☐ A $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.
☐ B $2^{\aleph_{\omega_1}} = \max \operatorname{pcf}\{\aleph_n : n \in \omega\}$.
☐ C $2^{\aleph_{\omega_1}} < \aleph_\gamma$, where $\gamma = (2^{\aleph_1})^+$.
☐ D $2^{\aleph_{\omega_1}} \leq \aleph_{\omega_3}$.

(10) One of the following statements is equivalent to the Continuum Hypothesis, which one?
☐ A $\mathbb{R}^2$ is the union of two sets $A$ and $B$ such that all vertical sections of $A$ and all horizontal sections of $B$ are countable.
☐ B $\mathbb{R}^2$ is the union of two sets $A$ and $B$ such that all vertical sections of $A$ and all horizontal sections of $B$ are finite.
☐ C There is a family of $\aleph_1$ many bounded sets in $\mathbb{R}$ such that every bounded set is a subset of one of them.
☐ D If $A \subset \mathbb{R}$ has cardinality less than $\mathbb{R}$; then $\mathbb{R} \setminus A$ contains a perfect subset.

(11) Which of the following statements is provable in $\mathsf{ZF}$ for all *cardinalities*?

☐ A $\kappa < \lambda$ implies $\kappa^\mu < \lambda^\mu$.

☐ B $\kappa < \lambda$ implies $\mu^\kappa < \mu^\lambda$.

☐ C $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

☐ D $\kappa < \lambda$ implies $\mu^\kappa < \mu^\lambda$.

(12) On the basis of $\mathsf{ZF}$ there are many equivalents of the Axiom of Choice. One of the following theorems of $\mathsf{ZFC}$ is equivalent to the Axiom of Choice. Which one?

☐ A The Hahn-Banach theorem.

☐ B Tychonoff's product theorem for Hausdorff spaces.

☐ C The maximal ideal theorem.

☐ D Tychonoff's product theorem for arbitrary topological spaces.

(13) We often use informal mathematical notation using curly braces to denote sets. However, not every expression corresponds to a set; sometimes, we denote proper classes. Among the following expressions, one corresponds to a proper class. Which one?

☐ A $\{x\ ;\ x$ is a nonempty subset of the natural numbers$\}$.

☐ B $\{x\ ;\ x$ is a finite set of real numbers$\}$.

☐ C $\{x\ ;\ x$ is a one-element set of complex numbers $\}$.

☐ D $\{x\ ;\ x$ is a two-element subset of a vector space$\}$.

(14) Suppose that there is a weakly compact cardinal kappa. Consider $\mathbf{V}_\kappa$ as a model of set theory. Only one of the following statements is not provably true in $\mathbf{V}_\kappa$:

☐ A The Axiom of Replacement.

☐ B For every $\alpha$, there is an inaccessible cardinal bigger than $\alpha$.

☐ C There is a weakly compact cardinal.

☐ D There is an inaccessible cardinal that is a limit of inaccessible cardinals.

(15) Which of the following is *not* provable in $\mathsf{ZF}$?

☐ A There is a surjection $f : \mathbb{R} \to \omega$.

☐ B There is a surjection $f : \mathbb{R} \to \omega_1$.

☐ C There is a surjection $f : \mathbb{R} \to \omega_2$.

☐ D There is an injection $f : \omega_1 \to \mathcal{P}(\mathbb{R})$.

(16) The regularity of successor cardinal numbers is a theorem of $\mathsf{ZFC}$, but cannot be proved in $\mathsf{ZF}$. In class we formulated the following instances of the Axiom of Choice; $\mathsf{AC}_X(Y)$ means: for every $f : X \to \mathcal{P}(Y) \setminus \{\emptyset\}$ there is $g : X \to Y$ such that $g(x) \in f(x)$ for all $x \in X$. Only one of the following instances of $\mathsf{AC}$ can, for a specific $\kappa$, prove that $\kappa^+$ is regular, which one?

☐ A $\mathsf{AC}_\kappa(\kappa^+)$.

☐ B $\mathsf{AC}_\kappa(\mathcal{P}(\kappa))$.

☐ C $\mathsf{AC}_\kappa(\mathbb{R})$.

☐ D $\mathsf{AC}_\kappa(\kappa)$.

(17) Which of the following partition relations is provable in $\mathsf{ZFC}$?

☐ A $5 \to (3)^2_2$.

☐ B $(2^{\aleph_0})^+ \to (\aleph_1)^2_2$.

☐ C $\aleph_1 \to (\aleph_1)^2_2$.

☐ D $2^{\aleph_0} \to (2^{\aleph_0})^2_2$.

(18) Only one of the following statements of cardinal arithmetic is provable in ZFC. Which one? ($\kappa$, $\lambda$ and $\mu$ are assumed to be infinite cardinal numbers.)

☐ A $(\kappa^+)^\lambda = \kappa^\lambda \cdot \kappa^+$.
☐ B if $\kappa \leq \lambda$ then $\kappa^\lambda = 2^\lambda$.
☐ C if $2^{\mathrm{cf}\,\kappa} < \kappa$ then $\kappa^{\mathrm{cf}\,\kappa} = \kappa^+$.
☐ D if $\mu < \kappa$ and $\mu^\lambda \geq \kappa$ then $\kappa^\lambda = \mu^\lambda$.

(19) Let $M$ be a countable elementary substructure of $H(\aleph_3)$. Which of the following statements is true?

☐ A $\omega_2 \subseteq M$.
☐ B $\mathcal{P}(\omega) \in M$.
☐ C $\omega_1 \cap M$ is transitive.
☐ D $\omega_2 \cap M$ is transitive.

(20) Which of the following statements is provable in ZFC?

☐ A There is a limit cardinal of cofinality $\aleph_\omega$.
☐ B There is a limit cardinal of cofinality $\aleph_4$.
☐ C If $\kappa$ is a regular limit cardinal then $\kappa = 2^{<\kappa}$.
☐ D There is a regular limit cardinal of cofinality $\aleph_3$.

(21) As in class we say that a theory $S$ is *stronger* than a theory $T$ is $S$ proves the consistency of $T$. Which of the following statements, when added to ZFC, yields the weakest theory.

☐ A There is one Mahlo cardinal.
☐ B There are two Mahlo cardinals.
☐ C There is an inaccessible cardinal that is a limit of Mahlo cardinals.
☐ D There is a proper class of inaccessible cardinals.

(22) Which of the following is provable in ZFC plus SCH (the Singular Cardinal Hypothesis):

☐ A If the GCH holds below $\aleph_{\omega_1}$ then $\aleph_{\omega_1}^{\aleph_0} = \aleph_{\omega_1+1}$.
☐ B If the GCH holds below $\aleph_{\omega_1}$ then then $\aleph_{\omega_1}^{\aleph_1} = \aleph_{\omega_1+1}$.
☐ C If $\aleph_{\omega_1}$ is a strong limit then $\aleph_{\omega_1}^{\aleph_1} = \aleph_{\omega_1}$.
☐ D If $\aleph_{\omega_1}$ is a strong limit then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+2}$.

(23) Who was the author of a book entitled *Paradoxien des Unendlichen* in which he disputed the rejection of the *actual infinite* that had had a dubious philosophical status since Aristotle's times?

☐ A Georg Cantor
☐ B Bernhard Bolzano
☐ C Ernst Zermelo
☐ D Johann von Neumann

(24) Let $\kappa$ be the *first* uncountable measurable cardinal number (we assume it exists); one of the following statements is not true, which:

☐ A Some $\sigma$-complete ultrafilter on $\kappa$ is $\kappa$-complete.
☐ B Every $\sigma$-complete ultrafilter on $\kappa$ is $\kappa$-complete.
☐ C $\kappa$ is a limit cardinal.
☐ D $\kappa$ is weakly compact.

(25) Which property characterizes weakly compact cardinals among the uncountable cardinals.
☐ A The tree property.
☐ B The property $\kappa \to (\kappa)^2_2$.
☐ C Weak compactness of $\mathcal{L}_{\kappa,\omega}$-languages.
☐ D Having a $\sigma$-complete ultrafilter.

**Exercise II** (15 points).

Prove that the Axiom of Replacement implies the Axiom of Separation (on the basis of the other axioms of set theory).

**Exercise III** (20 points).

Prove Kőnig's inequality: if $\kappa_i < \lambda_i$ for all $i \in I$, where the $\kappa_i$ and $\lambda_i$ are cardinal numbers, then

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$$

Specifically:

(1) Construct an explicit injection from $\bigcup_{i \in I}\{i\} \times \kappa_i$ into the product $\prod_{i \in I} \lambda_i$
(2) Show that there is no surjection from $\bigcup_{i \in I}\{i\} \times \kappa_i$ onto the product $\prod_{i \in I} \lambda_i$

**Exercise IV** (20 points).

Define a relation $\prec$ among ordered pairs of ordinals by $\langle \alpha, \beta \rangle \prec \langle \gamma, \delta \rangle$ if $\alpha + \beta < \gamma + \delta$ or $\alpha < \gamma$.

(1) Prove that $\prec$ well-orders the class of ordered pairs of ordinals and that for every $\langle \gamma, \delta \rangle$ the class $\{\langle \alpha, \beta \rangle : \langle \alpha, \beta \rangle \prec \langle \gamma, \delta \rangle\}$ is a set
(2) Prove that if $\kappa$ is a cardinal number then the order type of $\kappa \times \kappa$ with respect to $\prec$ is equal to $\kappa$.

**Exercise V** (29 points).

Remember that we said that $\mathcal{L}$ *satisfies the $\kappa$-weak compactness theorem* if for every $\mathcal{L}$-language $\mathcal{L}_S$, the following statement is true:

If $\Sigma$ is a set of $\mathcal{L}_S$-sentences such that $|\Sigma| \leq \kappa$ and every $\Sigma_0 \subseteq \Sigma$ with cardinality less than $\kappa$ has a model, then $\Sigma$ has a model.

(1) Give precise definitions of $\mathcal{L}_{\kappa\omega}$ and $\mathcal{L}_{\kappa\kappa}$ (4 points).
(2) Prove that if $\mathcal{L}_{\kappa\kappa}$ satisfies the $\kappa$-weak compactness theorem, then so does $\mathcal{L}_{\kappa\omega}$ (5 points).
(3) Consider the $\mathcal{L}_{\aleph_\omega\omega}$-language $\mathcal{L}^*$ with constants $\{c_\alpha \ ; \ \alpha \leq \aleph_\omega\}$ and a binary relation $R$. Construct a set of $\mathcal{L}^*$-sentences $\Sigma$ such that
    (a) $|\Sigma| \leq \aleph_\omega$,
    (b) if $\mathcal{M} = (M, x_\alpha, <) \models \Sigma$, then $\{x_{\aleph_n} \ ; \ n \in \omega\}$ is $<$-cofinal in $M$ (i.e., for every $x \in M$ there is an $n \in \omega$ such that $x < x_{\aleph_n}$), and
    (c) if $\mathcal{M} = (M, x_\alpha, <) \models \Sigma$, then
    if for all $\beta < \alpha$, $\mathcal{M} \models c_\beta \ R \ c_{\aleph_\omega}$, then $\mathcal{M} \models c_\alpha \ R \ c_{\aleph_\omega}$.
    Prove all claims about the properties of $\Sigma$. (15 points)
(4) Use the set $\Sigma$ constructed in (3) to show that $\aleph_\omega$ is not a weakly compact cardinal (5 points).