Mastermath Exam Set Theory
06-01-2017; 14:00–17:00.

This exam consists of multiple-choice questions, 1–12, and open questions, 13–16.
Record your answers to the multiple-choice questions in a readable table on the exam paper.

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(2) 1. Given our definitions of ordered pairs, $(x,y) = \{\{x\},\{x,y\}\}$, and natural numbers, $n = \{0,\ldots,n-1\}$, which of the following **is not** true:

A. $(0,1)\cap 3 = \emptyset$

B. $(0,0) = \{1\}$

C. $(0,1) = \{1,2\}$

D. $(0,1) = \bigcup(1,2)$

(2) 2. Which of the following *ordinal* inequalities **does** hold:

A. $2^\omega < 2^{2017\cdot\omega}$

B. $2\cdot\omega < 2017\cdot\omega$

C. $2^\omega < 2^{\omega\cdot 2017}$

D. $2+\omega < 2^{2017}+\omega$

(2) 3. Which of the following *cardinal* inequalities **does not** hold (in ZFC):

A. $\aleph_{2017}^{2017} < \aleph_{2018}$

B. $\aleph_\omega < \aleph_\omega^{\aleph_{2017}}$

C. $\aleph_{2018} \leqslant \aleph_{2017}^{\aleph_{2017}}$

D. $2^{\aleph_{2017}} < 2017^{\aleph_{2017}}$

(2) 4. Which of the following **is** a filter on $\mathbb{N}\times\mathbb{N}$?

A. $\{A\subseteq\mathbb{N}\times\mathbb{N} : \{n : (n,n)\notin A\} \text{ is finite}\}$

B. $\{A\subseteq\mathbb{N}\times\mathbb{N} : \{n : |\{m : (n,m)\notin A\}| < \aleph_0\} \text{ is finite}\}$

C. $\{A\subseteq\mathbb{N}\times\mathbb{N} : \{n : |\{m : (n,m)\notin A\}| = \aleph_0\} \text{ is finite}\}$

D. $\{A\subseteq\mathbb{N}\times\mathbb{N} : \{n : |\{m : (n,m)\notin A\}| \geqslant n!\} \text{ is finite}\}$

(2) 5. Assume $2^{\aleph_n} = \aleph_{\omega+n+2017}$ for $n \geqslant 2017$. Then the value of $2^{\aleph_\omega}$ is

A. still undetermined

B. smaller than $\aleph_{\omega+\omega}$

C. larger than $\aleph_{\omega+\omega}^{\aleph_0}$

D. equal to $\aleph_{\omega+\omega}^{\aleph_0}$

(2) 6. Which of the following statements **is not** provable in ZFC (κ, λ, and μ denote *infinite* cardinals):

A. If $\kappa\leqslant\lambda$ then $\kappa^\lambda > \aleph_0^\lambda$

B. $\aleph_{\alpha+2017}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta}\cdot\aleph_{\alpha+2017}$

C. If $\kappa<\lambda$ then $\kappa^\mu \leqslant \lambda^\mu$

D. If $\kappa<\lambda$ then $\mu^\kappa \leqslant \mu^\lambda$

More problems on the next page.

(2) 7. Which of the following statements **is not** provably equivalent to the Axiom of Choice in ZF.

A. For all sets X and Y we have $|X| \leqslant |Y|$ or $|Y| \leqslant |X|$

B. Every set has a linear order.

C. Zorn's Lemma

D. Every set has a well-order.

(2) 8. Let $\mathcal{U}$ be a free ultrafilter on ω. Which of the following families **is** an ultrafilter on ω.

A. $\{2A : A \in \mathcal{U}\}$, where $2A = \{2n : n \in A\}$

B. $\{A/2 : A \in \mathcal{U}\}$, where $A/2 = \{n : 2n \in A\}$

C. $\{A - 1 : A \in \mathcal{U}\}$, where $A - 1 = \{n : n + 1 \in A\}$

D. $\{{}^2\log A : A \in \mathcal{U}\}$, where ${}^2\log A = \{n : 2^n \in A\}$

(2) 9. Which of the following partition relations **is not** provable in ZFC:

A. $(2^{\aleph_{2017}})^+ \to (\aleph_{2018})^2_{2017}$

B. $\aleph_{2018} \to (\aleph_{2017})^2_{\aleph_{2017}}$

C. $\aleph_{2016} \to (\aleph_{2016}, \aleph_0)^2$

D. $2^{\aleph_{2017}} \nrightarrow (3)^2_{\aleph_{2017}}$

(2) 10. Let κ be a regular uncountable cardinal. Which of the following statements about cub and stationary subsets of κ **is** true.

A. The intersection of two stationary sets is again stationary.

B. The intersection of 2017 cub sets is again cub.

C. The union of κ many non-stationary sets is not stationary.

D. The intersection of a cub and a stationary set contains a cub set.

(2) 11. Which of the following statements **is not** true

A. Every weakly compact cardinal has the tree property.

B. Every weakly compact cardinal is a strong limit.

C. Every weakly compact cardinal is regular.

D. Every cardinal with the tree property is weakly compact.

(2) 12. Which of the following statements about the measurable cardinal κ **is not** true.

A. $\{\lambda < \kappa : \lambda \text{ is weakly compact}\}$ is stationary in κ.

B. There is a normal ultrafilter on κ.

C. If $2^\lambda = \lambda^+$ for all cardinals λ below κ then $2^\kappa = \kappa^+$.

D. The cardinal κ^+ is also measurable.

More problems on the next page.

13. *In this problem we do not assume the Axiom of Choice.* Recall that a set A is *finite* if there are $n \in \mathbb{N}$ and a bijection $f : n \to A$. Define A to be *D-finite* if every injective map $f : A \to A$ is surjective and *D-infinite* when it is not D-finite. Prove:

(7) a. (by induction) Every $n \in \mathbb{N}$ set is D-finite (hence every finite set is D-finite).

(4) b. $\mathbb{N}$ is D-infinite.

(7) c. For a set A the following are equivalent

(1) A is D-infinite

(2) $|A| + 1 = |A|$, i.e., there is a bijection $f : A \to A \cup \{p\}$, where $p \notin A$

(2) $|\mathbb{N}| \leqslant |A|$, i.e., there is an injection $f : \mathbb{N} \to A$

(16) 14. Prove the first non-trivial instance of the Erdős-Dushnik-Miller theorem:

$$\aleph_1 \to (\aleph_1, \aleph_0)^2$$

15. Let $f : \omega_1 \to \mathbb{R}$ be an injective map. For $q \in \mathbb{Q}$ put $A_q = \{\alpha : f(\alpha) < q\}$ and $B_q = \{\alpha : f(\alpha) > q\}$. Let $I = \{q : A_q \text{ contains a cub set}\}$ and $J = \{q : B_q \text{ contains a cub set}\}$.

(4) a. Prove: if $p \in I$ and $q \in J$ then $q < p$.

(4) b. Prove: $I \neq \mathbb{Q}$ and $J \neq \mathbb{Q}$.

(4) c. Prove: $\sup J < \inf I$ (by convention: $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$).

(4) d. Prove: there is a $q \in \mathbb{Q}$ such that both A_q and B_q are stationary.

(16) 16. Let κ be a measurable cardinal, with a normal ultrafilter $\mathcal{D}$, and let $\langle A_\alpha : \alpha < \kappa \rangle$ be a sequence of sets such that $A_\alpha \subseteq \alpha$ for all α. Prove that there is a subset A of κ such that $\{\alpha : A \cap \alpha = A_\alpha\}$ is stationary. *Hint*: Consider the partition $F : [\kappa]^2 \to \{0, 1\}$ defined by $F(\{\beta, \alpha\}) = 1$ if $A_\beta = A_\alpha \cap \beta$ and $F(\{\beta, \alpha\}) = 0$ if $A_\beta \neq A_\alpha \cap \beta$. Prove there is $X \in \mathcal{D}$ such that $F\big[[X]^2\big] = \{1\}$. To show that there is no $Y \in \mathcal{D}$ such that $F\big[[X]^2\big] = \{0\}$ you may use this special case of Exercise 10.6: if $f[\kappa]^2 \to \kappa$ if such that $f(x) < \min x$ whenever $\min x > 0$ then there is $Z \in \mathcal{D}$ such that f is constant on $[Z]^2$.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END