Mastermath Exam Set Theory
27-01-2017; 14:00–17:00.

This exam consists of multiple-choice questions, 1–12, and open questions, 13–17.
Record your answers to the multiple-choice questions in a readable table on the exam paper.

(2) 1. Given our definitions of ordered pairs, $(x, y) = \{\{x\}, \{x, y\}\}$), and natural numbers, $n = \{0, \dots, n-1\}$, which of the following **is true**:

A. $(0, 1) = \{1, 2\}$

B. $(0, 1) \in \{0, 1\}$

C. $(0, 1) \subseteq 2$

D. $0 \in (0, 1)$

(2) 2. Which of the following *ordinal* (in)equalities **does not hold**:

A. $2017^\omega = 2018^\omega$

B. $\omega \cdot 2017 < \omega \cdot 2018$

C. $2017^\omega < 2018^\omega$

D. $\omega + 2017 < \omega + 2018$

(2) 3. Which of the following *cardinal* (in)equalities **does hold** (in ZFC):

A. $2^{\aleph_{2017}} < \aleph_{2018}$

B. $\aleph_\omega = \aleph_\omega^{\aleph_{2017}}$

C. $\aleph_{\omega_1} < \aleph_{\omega_1}^{\aleph_1}$

D. $\aleph_\omega = \prod_{n<\omega} \aleph_n$

(2) 4. Which of the following **is not** a filter on $\mathbb{N} \times \mathbb{N}$?

A. $\{A \subseteq \mathbb{N} \times \mathbb{N} : \{n : (n, n) \notin A\} \text{ is finite}\}$

B. $\{A \subseteq \mathbb{N} \times \mathbb{N} : \{n : |\{m : (n, m) \notin A\}| < \aleph_0\} \text{ is finite}\}$

C. $\{A \subseteq \mathbb{N} \times \mathbb{N} : \{n : |\{m : (n, m) \notin A\}| = \aleph_0\} \text{ is finite}\}$

D. $\{A \subseteq \mathbb{N} \times \mathbb{N} : \{m : |\{n : (2n, m) \notin A\}| = \aleph_0\} \text{ is finite}\}$

(2) 5. Assume $2^{\aleph_n} = \aleph_{\omega+2017}$ for $n \geqslant 2017$. Then the value of $2^{\aleph_\omega}$ is

A. equal to $\aleph_{\omega+2017}$

B. smaller than $\aleph_{\omega+2017}$

C. larger than $\aleph_{\omega+2017}$

D. still undetermined

(2) 6. Which of the following statements **is** provable in ZFC (κ, λ, and μ denote *infinite* cardinals):

A. If $\kappa \leqslant \lambda$ then $\kappa^\lambda > \aleph_0^\lambda$

B. If $\kappa < \lambda$ then $\kappa^\mu < \lambda^\mu$

C. If $\kappa < \lambda$ then $\mu^\kappa < \mu^\lambda$

D. $\aleph_{\alpha+2017}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+2017}$

More problems on the next page.

(2) 7. Which of the following statements **is** provably equivalent to the Axiom of Choice in ZF.

A. Every set has a linear order

B. For all sets X and Y we have $|X| \leqslant |Y|$ or $|Y| \leqslant |X|$

C. Every filter on $\mathbb{N}$ can be extended to an ultrafilter

D. Every ideal in a ring with 1 can be extended to a maximal ideal

(2) 8. Let $\mathcal{U}$ be a free ultrafilter on ω. Which of the following families **is not** an ultrafilter on $\omega \times \omega$.

A. $\{X : \{n : (n, n) \in X\} \in \mathcal{U}\}$

B. $\{X : (\exists A \in \mathcal{U})(A \times A \subseteq X)\}$

C. $\{X : \{n : \{m : (n, m) \in X\} \in \mathcal{U}\} \in \mathcal{U}\}$

D. $\{X : \{n : (2017, n) \in X\} \in \mathcal{U}\}$

(2) 9. Which of the following partition relations **is** provable in ZFC:

A. $5 \to (3, 3)^2$

B. $\aleph_{2018} \to (\aleph_{2017})^2_{\aleph_{2017}}$

C. $\aleph_{2017} \to (\aleph_{2017}, \aleph_0)^2$

D. $2^{\aleph_{2017}} \to (3)^2_{\aleph_{2017}}$

(2) 10. Let κ be a regular uncountable cardinal. Which of the following statements about cub and stationary subsets of κ **is not true**.

A. The intersection of 2017 cub sets is again cub

B. The union of fewer than κ many non-stationary sets is non-stationary

C. The intersection of a cub and a stationary set is stationary

D. The intersection of two stationary sets is again stationary

(2) 11. Which of the following statements about weakly compact cardinals **is true**

A. Every weakly compact cardinal is measurable

B. The set of inaccessible cardinals below a weakly compact cardinal is closed and unbounded

C. If κ is weakly compact then $\kappa \to (\kappa)^{<\omega}$

D. Every weakly compact cardinal has the tree property

(2) 12. Which of the following statements about the measurable cardinal κ **is true**.

A. Every σ-complete ultrafilter on κ is κ-complete

B. $\{\lambda < \kappa : \lambda \text{ is weakly compact}\}$ is stationary in κ

C. Every ultrafilter on κ is σ-complete

D. The cardinal κ^+ is also measurable

More problems on the next page.

13. *In this problem we do not assume the Axiom of Choice.* Recall that a set A is *finite* if there are $n \in \mathbb{N}$ and a bijection $f : n \to A$. Define A to be *DD-finite* if there is a function $\sigma : A \to A$ such that the only σ-invariant subsets of A are $\emptyset$ and A itself. Note: X is σ-invariant if $\sigma[X] \subseteq X$.

(6) a. Prove (by induction) that every $n \in \mathbb{N}$ is DD-finite.

Now assume A is nonempty and DD-finite, witnessed by $\sigma : A \to A$. Define σ^n for $k \in \mathbb{N}$ by letting σ^0 be the identity map and, recursively, $\sigma^{k+1} = \sigma \circ \sigma^k$.

(4) b. Prove: if $x \in A$ then $\{\sigma^k(x) : k \geqslant 1\} = A$.

(6) c. Prove that A is finite. *Hint*: Take some $x \in A$ and prove there is an $n \in \mathbb{N}$ such that $A = \{\sigma^k(x) : k < n\}$.

14. The first non-trivial instance of the Erdős-Rado theorem states

$$(2^{\aleph_0})^+ \to (\aleph_1)^2_{\aleph_0}$$

(6) a. Explain what this statement means

(10) b. Prove the statement

15. Let L denote the set of Limit ordinals in ω_1. For every $\alpha \in L$ we choose an increasing sequence $\langle a(\alpha, n) : n < \omega \rangle$ of ordinals that converges to α.

(6) a. Prove: there is an n such that for *every* ordinal $\eta < \omega_1$ the set $\{\alpha \in L : a(\alpha, n) \geqslant \eta\}$ stationary. *Hint*: First show that for every η there is an $n = n(\eta)$ such that $\{\alpha \in L : a(\alpha, n) \geqslant \eta\}$ is stationary; then show that there is one n that works for all η simultaneously.

Define: $f : L \to \omega_1$ by $f(\alpha) = a(\alpha, n)$.

(6) b. Prove: for every $\eta < \omega_1$ there are a stationary set S and $\beta \geqslant \eta$ such that f is constant on S with value β.

(6) c. Prove that there is a pairwise disjoint family $\mathcal{S}$ of stationary subsets of ω_1 such that $|\mathcal{S}| = \aleph_1$.

(8) 16. Let κ be a weakly compact cardinal and let $(X, \prec)$ be a linearly ordered set of cardinality κ. Prove: X has a subset, H, of cardinality κ that is well-ordered by $\prec$ or $\succ$

(8) 17. Let κ be the first measurable cardinal. Let $\mathcal{D}$ be a normal ultrafilter on κ and let $j : V \to M$ be the corresponding elementary embedding. Prove that $j(\kappa)$ is measurable in M and that κ is not measurable in M.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END