Vector measures of bounded $\gamma$-variation and stochastic integrals

Jan van Neerven and Lutz Weis

Abstract. We introduce the class of vector measures of bounded $\gamma$-variation and study its relationship with vector-valued stochastic integrals with respect to Brownian motions.

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1. Introduction

It is well known that stochastic integrals can be interpreted as vector measures, the identification being given by the identity

$$F(A) = \int_A \phi dB.$$ 

Here, the driving process $B$ is a (semi)martingale (for instance, a Brownian motion), and $\phi$ is a stochastic process satisfying suitable measurability and integrability conditions. This observation has been used by various authors as the starting point of a theory of stochastic integration for vector-valued processes.

Let $X$ be a Banach space. In [5] we characterized the class of functions $\phi : (0, 1) \to X$ which are stochastically integrable with respect to a Brownian motion $(W_t)_{t \in [0,1]}$ as being the class of functions for which the operator $T_\phi : L^2(0,1) \to X$,

$$T_\phi f := \int_0^1 f(t)\phi(t) \, dt,$$

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belongs to the operator ideal $\gamma(L^2(0,1), X)$ of all $\gamma$-radonifying operators. Indeed, we established the Itô isomorphism

$$E\left\| \int_0^1 f \, dW \right\|^2 = \|Tf\|^2_{\gamma(L^2(0,1), X)}.$$  

The linear subspace of all operators in $\gamma(L^2(0,1), X)$ of the form $T = Tf$ for some function $f : (0, 1) \to X$ is dense, but unless $X$ has cotype 2 it is strictly smaller than $\gamma(L^2(0,1), X)$. This means that in general there are operators $T \in \gamma(L^2(0,1), X)$ which are not representable by an $X$-valued function. Since the space of test functions $\mathscr{D}(0,1)$ embeds in $L^2(0,1)$, by restriction one could still think of such operators as $X$-valued distributions. It may be more intuitive, however, to think of $T$ as an $X$-valued vector measure. We shall prove (see Theorem 2.3 and the subsequent remark) that if $X$ does not contain a closed subspace isomorphic to $c_0$, then the space $\gamma(L^2(0,1), X)$ is isometrically isomorphic in a natural way to the space of $X$-valued vector measures on $(0,1)$ which are of bounded $\gamma$-variation.

This gives a ‘measure theoretic’ description of the class of admissible integrands for stochastic integrals with respect to Brownian motions. The condition $c_0 \not\subseteq X$ can be removed if we replace the space of $\gamma$-radonifying operators by the larger space of all $\gamma$-summing operators (which contains the space of all $\gamma$-radonifying operators isometrically as a closed subspace).

Vector measures of bounded $\gamma$-variation behave quite differently from vector measures of bounded variation. For instance, the question whether an $X$-valued vector measure of bounded $\gamma$-variation can be represented by an $X$-valued function is not linked to the Radon-Nikodým property, but rather to the type 2 and cotype 2 properties of $X$ (see Corollaries 2.5 and 2.6).

In section 3 we consider yet another class of vector measures whose variation is given by certain random sums, and we show that a function $\phi : (0, 1) \to X$ is stochastically integrable with respect to a Brownian motion $(W_t)_{t \in [0,1]}$ on a probability space $(\Omega, \mathcal{F}, P)$ if and only if the formula $F(A) := \int_A \phi \, dW$ defines an $L^2(\Omega; X)$-valued vector measure $F$ in this class.

2. Vector measures of bounded $\gamma$-variation

Let $(S, \Sigma)$ be a measurable space, $X$ a Banach space, and $(\gamma_n)_{n \geq 1}$ a sequence of independent standard Gaussian random variables defined on a probability space $(\Omega, \mathcal{F}, P)$.

**Definition 2.1.** We say that a countably additive vector measure $F$ has bounded $\gamma$-variation with respect to a probability measure $\mu$ on $(S, \Sigma)$ if $\|F\|_{V_\gamma(\mu; X)} < \infty$, where

$$\|F\|_{V_\gamma(\mu; X)} := \sup \left( E\left\| \sum_{n=1}^N \gamma_n \frac{F(A_n)}{\sqrt{\mu(A_n)}} \right\|^2 \right)^{1/2},$$

the supremum being taken over all finite collections of disjoint sets $A_1, \ldots, A_N \in \Sigma$ such that $\mu(A_n) > 0$ for all $n = 1, \ldots, N$. 

It is routine to check (e.g. by an argument similar to [4, Proposition 5.2]) that the space \( V_\gamma(\mu; X) \) of all countably additive vector measures \( F : \Sigma \to X \) which have bounded \( \gamma \)-variation with respect to \( \mu \) is a Banach space with respect to the norm \( \| \cdot \|_{V_\gamma(\mu; X)} \). Furthermore, every vector measure which is of bounded \( \gamma \)-variation is of bounded 2-semivariation.

In order to give a necessary and sufficient condition for a vector measure to have bounded \( \gamma \)-variation we need to introduce the following terminology. A bounded operator \( T : H \to X \), where \( H \) is a Hilbert space, is said to be \( \gamma \)-summing if there exists a constant \( C \) such that for all finite orthonormal systems \( \{ h_1, \ldots, h_N \} \) in \( H \) one has

\[
E \left\| \sum_{n=1}^N \gamma_n Th_n \right\|^2 \leq C^2.
\]

The least constant \( C \) for which this holds is called the \( \gamma \)-summing norm of \( T \), notation \( \| T \|_{\gamma,\infty(H,X)} \). With respect to this norm, the space \( \gamma,\infty(H,X) \) of all \( \gamma \)-summing operators from \( H \) to \( X \) is a Banach space which contains all finite rank operators from \( H \) to \( X \). In what follows we shall make free use of the elementary properties of \( \gamma \)-summing operators. For a systematic exposition of these we refer to [2, Chapter 12] and the lecture notes [4].

**Theorem 2.2.** Let \( \mathcal{A} \) be an algebra of subsets of \( S \) which generates the \( \sigma \)-algebra \( \Sigma \), and let \( F : \mathcal{A} \to X \) be a finitely additive mapping. If, for some \( 1 \leq p < \infty \), \( T : L^p(\mu) \to X \) is a bounded operator such that

\[
F(A) = T1_A, \quad A \in \mathcal{A},
\]

then \( F \) has a unique extension to a countably additive vector measure on \( \Sigma \) which is absolutely continuous with respect to \( \mu \). If \( T : L^2(\mu) \to X \) is \( \gamma \)-summing, then the extension of \( F \) has bounded \( \gamma \)-variation with respect to \( \mu \) and we have

\[
\| F \|_{V_\gamma(\mu; X)} \leq \| T \|_{\gamma,\infty(L^2(\mu); X)}.
\]

**Proof.** We define the extension \( F : \Sigma \to X \) by \( F(A) := T1_A, \quad A \in \Sigma \). To see that \( F \) is countably additive, consider a disjoint union \( A = \bigcup_{n \geq 1} A_n \) with \( A_n, A \in \Sigma \). Then \( \lim_{N \to \infty} \bigcup_{n=1}^N A_n = 1_A \) in \( L^p(\mu) \) and therefore

\[
\lim_{N \to \infty} \sum_{n=1}^N F(A_n) = \lim_{N \to \infty} T \sum_{n=1}^N 1_A = T1_A = F(A).
\]

The absolute continuity of \( F \) is clear. To prove uniqueness, suppose \( \tilde{F} : \Sigma \to X \) is another countably additive vector measure extending \( F \). For each \( x^* \in X^* \), \( \langle \tilde{F}, x^* \rangle \) and \( \langle F, x^* \rangle \) are finite measures on \( \Sigma \) which agree on \( \mathcal{A} \), and therefore by Dynkin’s lemma they agree on all of \( \Sigma \). This being true for all \( x^* \in X^* \), it follows that \( \tilde{F} = F \) by the Hahn-Banach theorem.

Suppose next that \( T : L^2(\mu) \to X \) is \( \gamma \)-summing, and consider a finite collection of disjoint sets \( A_1, \ldots, A_N \) in \( \Sigma \) such that \( \mu(A_n) > 0 \) for all \( n = 1, \ldots, N \).
The functions \( f_n = 1_{A_n}/\sqrt{\mu(A_n)} \) are orthonormal in \( L^2(\mu) \) and therefore
\[
E \left\| \sum_{n=1}^{N} \gamma_n \frac{F(A_n)}{\sqrt{\mu(A_n)}} \right\|^2 = E \left\| \sum_{n=1}^{N} \gamma_n T f_n \right\|^2 \leq \|T\|^2_{\gamma_n(L^2(\mu),X)}.
\]

It follows that \( F \) has bounded \( \gamma \)-variation with respect to \( \mu \) and that \( \|F\|_{V_\gamma(\mu;X)} \leq \|T\|_{\gamma_n(L^2(\mu),X)} \).

**Theorem 2.3.** For a countably additive vector measure \( F : \Sigma \to X \) the following assertions are equivalent:

1. \( F \) has bounded \( \gamma \)-variation with respect to \( \mu \);
2. There exists a \( \gamma \)-summing operator \( T : L^2(\mu) \to X \) such that \( F(A) = T1_A, \ A \in \Sigma. \)

In this situation we have
\[
\|F\|_{V_\gamma(\mu;X)} = \|T\|_{\gamma_n(L^2(\mu),X)}.
\]

**Proof.** (1) \( \Rightarrow \) (2): Suppose that \( F \) has bounded \( \gamma \)-variation with respect to \( \mu \). For a simple function \( f = \sum_{n=1}^{N} c_n 1_{A_n} \), where the sets \( A_n \in \Sigma \) are disjoint and of positive \( \mu \)-measure, define
\[
T f := \sum_{n=1}^{N} c_n F(A_n).
\]

By the Cauchy-Schwarz inequality, for all \( x^* \in X^* \) we have
\[
|\langle T f, x^* \rangle| = \left| E \sum_{n=1}^{N} \gamma_n c_n \sqrt{\mu(A_n)} \cdot \sum_{n=1}^{N} \gamma_n \frac{\langle F(A_n), x^* \rangle}{\sqrt{\mu(A_n)}} \right|
\leq \left( E \left\| \sum_{n=1}^{N} \gamma_n c_n \sqrt{\mu(A_n)} \right\|^2 \right)^{\frac{1}{2}} \left( E \left\| \sum_{n=1}^{N} \gamma_n \frac{\langle F(A_n), x^* \rangle}{\sqrt{\mu(A_n)}} \right\|^2 \right)^{\frac{1}{2}}
\leq \left( \sum_{n=1}^{N} |c_n|^2 \mu(A_n) \right)^{\frac{1}{2}} \|F\|_{V_\gamma(\mu;X)} \|x^*\|
= \|f\|_{L^2(\mu)} \|F\|_{V_\gamma(\mu;X)} \|x^*\|.
\]

It follows that \( T \) is bounded and \( \|T\|_{\mathcal{L}(L^2(\mu),X)} \leq \|F\|_{V_\gamma(\mu;X)}. \) To prove that \( T \) is \( \gamma \)-summing we shall first make the simplifying assumption that the \( \sigma \)-algebra \( \Sigma \) is countably generated. Under this assumption there exists an increasing sequence of finite \( \sigma \)-algebras \( (\Sigma_n)_{n \geq 1} \) such that \( \Sigma = \bigvee_{n \geq 1} \Sigma_n \). Let \( P_n \) be the orthogonal projection in \( L^2(\mu) \) onto \( L^2(\Sigma_n, \mu) \) and put \( T_n := T \circ P_n \). These operators are of finite rank and we have \( \lim_{n \to \infty} T_n \to T \) in the strong operator topology of \( \mathcal{L}(L^2(\mu),X) \).

Fix an index \( n \geq 1 \) for the moment. Since \( \Sigma_n \) is finitely generated there exists a partition \( S = \bigcup_{j=1}^{N} A_j \), where the disjoint sets \( A_1, \ldots, A_N \) generate \( \Sigma_n \). Assuming that \( \mu(A_j) > 0 \) for all \( j = 1, \ldots, M \) and \( \mu(A_j) = 0 \) for \( j = M+1, \ldots, N \),
the functions \( g_j = 1_{A_j}/\sqrt{\mu(A_j)} \), \( j = 1, \ldots, M \), form an orthonormal basis for \( L^2(\Sigma, \mu) \) and

\[
\|T_n\|_{\gamma(L^2(\mu), X)}^2 = \|T_n\|_{\gamma(L^2(\Sigma, \mu), X)}^2 = E \left\| \sum_{j=1}^{M} \gamma_j T g_j \right\|^2 = E \left\| \sum_{j=1}^{M} \gamma_j \frac{F(A_j)}{\sqrt{\mu(A_n)}} \right\|^2 \leq \|F\|_{\gamma(\mu; X)}^2,
\]

the first identity being a consequence of [4, Corollary 5.5] and the second of [4, Lemma 5.7]. It follows that the sequence \( (T_n)_{n \geq 1} \) is bounded in \( \gamma(L^2(\mu), X) \). By the Fatou lemma, if \( \{f_1, \ldots, f_k\} \) is an orthonormal family in \( L^2(\mu) \), then

\[
E \left\| \sum_{j=1}^{k} \gamma_j T f_j \right\|^2 \leq \liminf_{n \to \infty} E \left\| \sum_{j=1}^{k} \gamma_j T_n f_j \right\|^2 \leq \|T_n\|_{\gamma(L^2(\mu), X)}^2 \leq \|F\|_{\gamma(\mu; X)}^2.
\]

This proves that \( T \) is \( \gamma \)-summing and \( \|T\|_{\gamma(L^2(\mu), X)} \leq \|F\|_{\gamma(\mu; X)} \).

It remains to remove the assumption that \( \Sigma \) is countably generated. The preceding argument shows that if we define \( T \) in the above way, then its restriction to \( L^2(\Sigma', \mu) \) is \( \gamma \)-summing for every countably generated \( \sigma \)-algebra \( \Sigma' \subseteq \Sigma \), with a uniform bound

\[
\|T\|_{\gamma(L^2(\Sigma', \mu), X)} \leq \|F\|_{\gamma(\mu; X)}.
\]

Since every finite orthonormal family \( \{f_1, \ldots, f_k\} \) in \( L^2(\mu) \) is contained in \( L^2(\Sigma', \mu) \) for some countably generated \( \sigma \)-algebra \( \Sigma' \subseteq \Sigma \), we see that

\[
E \left\| \sum_{j=1}^{k} \gamma_j T f_j \right\|^2 \leq \|T\|_{\gamma(L^2(\Sigma', \mu), X)}^2 \leq \|F\|_{\gamma(\mu; X)}^2.
\]

It follows that \( T \) is \( \gamma \)-summing and \( \|T\|_{\gamma(L^2(\mu), X)} \leq \|F\|_{\gamma(\mu; X)} \).

(2)\(\Rightarrow\)(1): This implication is contained in Theorem 2.2.

By a theorem of Hoffmann-Jørgensen and Kwapień [3, Theorem 9.29], if \( X \) is a Banach space not containing an isomorphic copy of \( c_0 \), then for any Hilbert space \( H \) one has

\[
\gamma(\infty)(H, X) = \gamma(H, X),
\]

where by definition \( \gamma(H, X) \) denotes the closure in \( \gamma(\infty)(H, X) \) of the finite rank operators from \( H \) to \( X \). Since any operator in this closure is compact we obtain:

**Corollary 2.4.** If \( X \) does not contain an isomorphic copy of \( c_0 \) and \( F : \Sigma \to X \) has bounded \( \gamma \)-variation with respect to \( \mu \), then \( F \) has relatively compact range.

Using the terminology of [5], a theorem of Rosiński and Suchanecki [6] asserts that if \( X \) has type 2 we have a continuous inclusion \( L^2(\mu; X) \hookrightarrow \gamma(L^2(\mu), X) \) and that if \( X \) has cotype 2 we have a continuous inclusion \( \gamma(\infty)(L^2(\mu), X) \hookrightarrow L^2(\mu; X) \). In both cases the embedding is contractive, and the relation between the operator \( T \) and the representing function \( \phi \) is given by

\[
Tf = \int_S f \phi \, d\mu, \quad f \in L^2(\mu).
\]
If \( \dim L^2(\mu) = \infty \), then in the converse direction the existence of a continuous embedding \( L^2(\mu; X) \hookrightarrow \gamma_\infty(L^2(\mu), X) \) (respectively \( \gamma(L^2(\mu), X) \hookrightarrow L^2(\mu; X) \)) actually implies the type 2 property (respectively the cotype 2 property) of \( X \).

**Corollary 2.5.** Let \( X \) have type 2. For all \( \phi \in L^2(\mu; X) \) the formula

\[
F(A) := \int_A \phi \, d\mu, \quad A \in \Sigma,
\]

defines a countably additive vector measure \( F : \Sigma \to X \) which has bounded \( \gamma \)-variation with respect to \( \mu \). Moreover,

\[
\|F\|_{V^\gamma(\mu; X)} \leq \|\phi\|_{L^2(\mu; X)}.
\]

If \( \dim L^2(\mu) = \infty \), this property characterises the type 2 property of \( X \).

**Proof.** By the theorem of Rosiński and Suchanecki, \( \phi \) represents an operator \( T \in \gamma(L^2(\mu), X) \) such that \( T1_A = \int_A \phi \, d\mu = F(A) \) for all \( A \in \Sigma \). The result now follows from Theorem 2.2. The converse direction follows from Theorem 2.3 and the preceding remarks. \( \Box \)

**Corollary 2.6.** Let \( X \) have cotype 2. If \( F : \Sigma \to X \) has bounded \( \gamma \)-variation with respect to \( \mu \), there exists a function \( \phi \in L^2(\mu; X) \) such that

\[
F(A) = \int_A \phi \, d\mu, \quad A \in \Sigma.
\]

Moreover,

\[
\|\phi\|_{L^2(\mu; X)} \leq \|F\|_{V^\gamma(\mu; X)}.
\]

If \( \dim L^2(\mu) = \infty \), this property characterises the cotype 2 property of \( X \).

**Proof.** By Theorem 2.3 there exists an operator \( T \in \gamma_\infty(L^2(\mu), X) \) such that \( F(A) = T1_A \) for all \( A \in \Sigma \). Since \( X \) has cotype 2, \( X \) does not contain an isomorphic copy of \( c_0 \) and therefore the theorem of Hoffmann-Jørgensen and Kwapień implies that \( T \in \gamma(L^2(\mu), X) \). Now the theorem of Rosiński and Suchanecki shows that \( T \) is represented by a function \( \phi \in L^2(\mu; X) \). The converse direction follows from Theorem 2.2 and the remarks preceding Corollary 2.5. \( \Box \)

### 3. Vector measures of bounded randomised variation

Let \( (S, \Sigma) \) be a measurable space and \( (r_n)_{n \geq 1} \) a Rademacher sequence, i.e., a sequence of independent random variables with \( P(r_n = \pm 1) = \frac{1}{2} \).

**Definition 3.1.** A countably additive vector measure \( F : \Sigma \to X \) is of bounded randomised variation if \( \|F\|_{V^r(\mu; X)} < \infty \), where

\[
\|F\|_{V^r(\mu; X)} = \sup \left( \mathbb{E} \left( \left\| \sum_{n=1}^N r_n F(A_n) \right\|^2 \right)^{\frac{1}{2}} \right),
\]

the supremum being taken over all finite collections of disjoint sets \( A_1, \ldots, A_N \in \Sigma \).
Clearly, if $F$ is of bounded variation, then $F$ is of bounded randomised variation. The converse fails; see Example 1. If $X$ has finite cotype, standard comparison results for Banach space-valued random sums [2, 3] imply that an equivalent norm is obtained when the Rademacher variables are replaced by Gaussian variables.

It is routine to check that the space $V^r(\mu; X)$ of all countably additive vector measures $F : \Sigma \to X$ of bounded randomised variation is a Banach space with respect to the norm $\| \cdot \|_{V^r(\mu; X)}$.

In Theorem 3.2 below we establish a connection between measures of bounded randomised variation and the theory of stochastic integration. For this purpose we need the following terminology. A Brownian motion on $(\Omega, F, P)$ indexed by another probability space $(S, \Sigma, \mu)$ is a mapping $W : \Sigma \to L^2(\Omega)$ such that:

(i) For all $A \in \Sigma$ the random variable $W(A)$ is centred Gaussian with variance $\mathbb{E}(W(A))^2 = \mu(A)$;

(ii) For all disjoint $A, B \in \Sigma$ the random variables $W(A)$ and $W(B)$ are independent.

A strongly $\mu$-measurable function $\phi : S \to X$ is stochastically integrable with respect to $W$ if for all $x^* \in X^*$ we have $\langle \phi, x^* \rangle \in L^2(\mu)$ (i.e. $f$ belongs to $L^2(\mu)$ scalarly) and for all $A \in \Sigma$ there exists a strongly measurable random variable $Y_A : \Omega \to X$ such that for all $x^* \in X^*$ we have

$$\langle Y_A, x^* \rangle = \int_A \langle \phi, x^* \rangle \, dW$$

almost surely. Note that each $Y_A$ is centred Gaussian and therefore belongs to $L^2(\Omega; X)$ by Fernique’s theorem; the above equality then holds in the sense of $L^2(\Omega)$. We define the stochastic integral of $\phi$ over $A$ by $\int_A \phi \, dW := Y_A$. For more details and various equivalent definitions we refer to [5].

**Theorem 3.2.** Let $W : \Sigma \to L^2(\Omega)$ be a Brownian motion. For a strongly $\mu$-measurable function $\phi : S \to X$ the following assertions are equivalent:

1. $\phi$ is stochastically integrable with respect to $W$;
2. $\phi$ belongs to $L^2(\mu)$ scalarly and there exists a countably additive vector measure $F : \Sigma \to X$, of bounded $\gamma$-variation with respect to $\mu$, such that for all $x^* \in X^*$ we have

$$\langle F(A), x^* \rangle = \int_A \langle \phi, x^* \rangle \, d\mu, \quad A \in \Sigma;$$

3. $\phi$ belongs to $L^2(\mu)$ scalarly and there exists a countably additive vector measure $G : \Sigma \to L^2(\Omega; X)$ of bounded randomised variation such that for all $x^* \in X^*$ we have

$$\langle G(A), x^* \rangle = \int_A \langle \phi, x^* \rangle \, dW, \quad A \in \Sigma.$$
In this situation we have

\[ \|F\|_{V, (\mu; X)} = \|G\|_{V^*(\mu; L^2(\Omega; X))} = \left( E \left\| \int_S \phi \, dW \right\|^2 \right)^{\frac{1}{2}}. \]

**Proof.** (1)⇔(2): This equivalence is immediate from Theorem 2.3 and the fact, proven in [5], that \( \phi \) is stochastically integrable with respect to \( W \) if and only if there exists an operator \( T \in \gamma(L^2(\mu), X) \) such that

\[ Tf = \int_S f \, \phi \, d\mu, \quad f \in L^2(\mu). \]

In this case we also have

\[ \|T\|_{\gamma(L^2(\mu), X)} = \left( E \left\| \int_S \phi \, dW \right\|^2 \right)^{\frac{1}{2}}. \]

In view of Theorem 2.3, this proves the identity

\[ \|F\|_{V, (\mu; X)} = \left( E \left\| \int_S \phi \, dW \right\|^2 \right)^{\frac{1}{2}}. \]

(1)⇒(3): Define \( G : \Sigma \to L^2(\Omega; X) \) by

\[ G(A) := \int_A \phi \, dW, \quad A \in \Sigma. \]

By the \( \gamma \)-dominated convergence theorem [5], \( G \) is countably additive. To prove that \( G \) is of bounded randomised variation we consider disjoint sets \( A_1, \ldots, A_N \in \Sigma \). If \( (\tilde{r}_n)_{n \geq 1} \) is a Rademacher sequence on a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\), then by randomisation we have

\[
\tilde{E} \left\| \sum_{n=1}^N \tilde{r}_n G(A_n) \right\|_{L^2(\Omega; X)}^2 = \tilde{E}E \left\| \sum_{n=1}^N \tilde{r}_n \int_{A_n} \phi \, dW \right\|^2 = E \left\| \sum_{n=1}^N \int_{A_n} \phi \, dW \right\|^2 \leq E \left\| \int_S \phi \, dW \right\|^2.
\]

with equality if \( \bigcup_{n=1}^N A_n = S \). In the second identity we used that the \( X \)-valued random variables \( \int_{A_n} \phi \, dW \) are independent and symmetric. The final inequality follows by, e.g., covariance domination [5] or an application of the contraction principle. It follows that \( G \) is a countably additive vector measure of bounded randomised variation and

\[ \|G\|_{V^*(\mu; X)} = \left( E \left\| \int_S \phi \, dW \right\|^2 \right)^{\frac{1}{2}}. \]

(3)⇒(1): This is immediate from the definition of stochastic integrability. \( \square \)

**Example 1.** If \( W \) is a standard Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\) indexed by the Borel interval \(([0, 1], \mathcal{B}, m)\), then \( W \) is a countably additive vector measure with values in \( L^2(\Omega) \) which is of bounded randomised variation, but of unbounded variation. The first claim follows from Theorem 3.2 since \( W(A) = \int_A 1 \, dW \) for all
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Borel sets $A$. To see that $W$ is of unbounded variation, note that for any partition $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$ we have

$$\sum_{n=1}^{N} \|W((t_{n-1}, t_n))\|_{L^2(\Omega)} = \sum_{n=1}^{N} \sqrt{t_n - t_{n-1}}.$$  

The supremum over all possible partitions of $[0, 1]$ is unbounded.

References


Jan van Neerven  
Delft University of Technology  
Delft Institute of Applied Mathematics  
P.O. Box 5031, 2600 GA Delft  
The Netherlands  
e-mail: J.M.A.M.vanNeerven@TUDelft.nl

Lutz Weis  
University of Karlsruhe  
Mathematisches Institut I  
D-76128 Karlsruhe  
Germany  
e-mail: Lutz.Weis@math.uni-karlsruhe.de