Contractive semigroups with nonlocal conditions

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A problem of existence of contractive positive semigroups generated by elliptic operators with nonlocal conditions arises in the theory of diffusion processes (see [1]). This problem was studied in the case of differential operators of the second order in [2].
Consider an unbounded integro-differential operator of the form

$$Au(x) = \triangle u(x) + \int\limits_{Q_x} [u(x+z) - u(x) - \nabla u(x) \cdot z] m(dz) \qquad (x \in Q),$$

where $Q \subset \mathbb{R}^n$ is a bounded domain, $\partial Q \in C^\infty$, $Q_x = \{z \in \mathbb{R}^n : x + z \in \overline{Q}\}$, and $m(\cdot)$ is a nonnegative Borel measure on Q_x such that

$$\int\limits_{Q_x \cap \{|z| \le r\}} |z|^2 m(dz) \to 0 \quad (r \to 0), \qquad \int\limits_{Q_x \cap \{|z| > r\}} |z| m(dz) < \infty \quad (r > 0).$$

Consider a nonlocal boundary condition of the form

$$Bu(x) = \gamma(x) u(x) + \int\limits_{\overline{Q}} [u(x) - u(y)] \mu(x, dy) = 0 \qquad (x \in \partial Q),$$

where $\gamma(x) \ge 0$, $\mu(x, \cdot)$ is a nonnegative Borel measure on $\overline{Q}$.
Denote $C_B(\overline{Q}) = \{u \in C(\overline{Q}) : Bu = 0\}$. Define an operator $A_B : C_B(\overline{Q}) \to C_B(\overline{Q})$ by the formula $A_B u = Au \quad (u \in \mathcal{D}(A_B)), \quad \mathcal{D}(A_B) = \{u \in C^2(Q) \cap C_B(\overline{Q}) : Au \in C_B(\overline{Q})\}$.

If the measure $\mu(x, \cdot)$ satisfies certain geometrical conditions formulated in [2], then the following result is valid.

Theorem. *The operator $\overline{A}_B : C_B(\overline{Q}) \to C_B(\overline{Q})$ is the infinitesimal generator of a contractive positive semigroup on $C_B(\overline{Q})$.*

[1] Taira K. Diffusion Processes and Partial Differential Equations, New York, London, Academic Press, 1988.

[2] Galakhov E.I., Skubachevskii A.L., Matematicheskii Sbornik **165 (207)** (1998), 45–78; English transl. to appear in Math. Sb.