The semistatic limit for Maxwell's equations in an exterior domain

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In this poster a L^p-theory for the Maxwell-system in the semistatic limit case in an exterior domain is described. The problem under consideration is of mixed type, since the possibly nonlinear electric conductivity vanishes on a certain subset of the domain. The solution is obtained from a singular perturbation of the full Maxwell-system including the displacement current, that is

$$\text{curl } \mathbf{h}_\varepsilon = \varepsilon \partial_t \mathbf{E}_\varepsilon + \mathbf{S}(\mathbf{E}_\varepsilon) + \mathbf{j}_0 \tag{1}$$

$$\text{curl } \mathbf{E}_\varepsilon = -\mu \partial_t \mathbf{h}_\varepsilon \tag{2}$$

supplemented by the initial-boundary-conditions

$$\vec{n} \wedge \mathbf{E}_\varepsilon = 0 \text{ on } [0,\infty) \times \partial\Omega, \tag{3}$$

$$\mathbf{E}_\varepsilon(0) = \mathbf{E}_0 \text{ and } \mathbf{h}_\varepsilon(0) = \mathbf{h}_0. \tag{4}$$

Here $\Omega \subset \mathbb{R}^3$ is a domain with bounded complement, $\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon$ denote the electric and magnetic field respectively which depend on the time $t \geq 0$ and the space-variable $x \in \Omega$. On the perfectly conducting boundary $\partial\Omega$ the tangential-component of $\mathbf{E}$ must vanish, which is expressed by boundary condition 3. $\mathbf{j}_0$ is a prescribed external current, which is assumed to be divergence-free. The charge current $\mathbf{S}(\mathbf{E}_\varepsilon)$ may depend nonlinearely and nonlocally on the electric field, such that

$$\mathbf{S}(\mathbf{E})(t,x) = 0 \text{ if } x \in \Omega_0 = \Omega \setminus \overline{G}, \tag{5}$$

with some bounded set $G \subset \Omega$.

$$\mathbf{S} : L^{\tilde{r}}_{loc}([0,\infty), \mathcal{G}) \to L^{\tilde{r}^*}_{loc}([0,\infty), L^2(\Omega) \cap \mathcal{G}^*) \text{ with some } \tilde{r} \in [1,\infty) \tag{6}$$

is a monotone operator, which is causal in the sense that

$$\mathbf{S}(\mathbf{E})(t,x) = \mathbf{S}(\tilde{\mathbf{E}})(t,x) \text{ for all } t \in [0,T] \tag{7}$$

provided that $\mathbf{E}(t,x) = \tilde{\mathbf{E}}(t,x)$ for all $t \in [0,T]$.
In 6 it is assumed that $L^2(G) \subset \mathcal{G} \subset L^{r_0}(G)$, where $r_0 \in [6/5, 2]$ dependes on Ω_0.
The assumptions on $\mathbf{S}$ are fulfilled in the case that

$$\mathbf{S}(\mathbf{E})(t,x) = \sigma(x, |\mathbf{E}(t,x)|)\mathbf{E}(t,x) \tag{8}$$

with a field-dependent electric conductivity $\sigma(x,u) = \sigma_0(x)(1+u)^{\tilde{r}-2}$.
Here $\tilde{r} \in [r_0, 2]$ and $\sigma_0 \in L^\infty(G)$ with $\sigma_0 > 0$, which may have mild degeneracies on G with respect to x, i.e. it is not assumed that σ_0 is uniformly positive. More precisely,
$\sigma_0^{-1} \in L^{r_0/(\tilde{r}-r_0)}(G)$, where $6/5 < r_0 \leq 2$ as above.
In particular, the linear case $\tilde{r} = 2$, i.e. $\sigma(x,u) = \sigma_0(x)$ is possible.

In 2 $\mu \in L^\infty(\mathbb{R}^3)$ is the magnetic susceptibility, which is assumed to be uniformly positive. The dielectric susceptibility $\varepsilon > 0$ is considered as a parameter in the above system.
In many situations the displacement current $\varepsilon \partial_t \mathbf{E}_\varepsilon$ is small in comparison with the charge currents and is often neglected. Therefore, it is the aim of this paper to investigate the singular limit $\varepsilon \to 0$. By letting $\varepsilon \to 0$ one obtains fomally from 1-4 the quasi-stationary Maxwell-System

$$\text{curl } \mathbf{h} = \mathbf{S}(\mathbf{E}) + \mathbf{j}_0 \tag{9}$$

$$\text{curl } \mathbf{E} = -\mu \partial_t \mathbf{h} \tag{10}$$

$$\text{div } \mathbf{E} = 0 \text{ on } [0,\infty) \times \Omega_0 \tag{11}$$

supplemented by the initial-boundary-conditions

$$\vec{n} \wedge \mathbf{E} = 0 \text{ on } [0,\infty) \times \partial\Omega \tag{12}$$

$$\text{and } \mathbf{h}(0) = \mathbf{h}_0 \text{ on } \Omega \tag{13}$$

$$\int_{\partial \mathcal{C}_k} \vec{n}\mathbf{E}(t,x) dS = Q_k \text{ for all } t \in [0,\infty), k \in \{1,..,N\}. \tag{14}$$

Here $\mathcal{C}_k$,$k = 1,..,N$ denote the connected components of $\mathcal{C} \stackrel{\text{def}}{=} \mathbb{R}^3 \setminus \overline{\Omega_0}$, the complement of the 'vacuum-region' Ω_0. The physical meaning of 14 is that the total electric charge Q_k on each $\mathcal{C}_k$ is prescribed. This problem has been investigated in [4] in the case that the domain Ω is bounded and σ is linear with respect to $\mathbf{E}$ and uniformly positive. See also [1] and [2], where a temperature dependent electrical conductivity is considered. However, in this paper Ω is an exterior domain and the electrical conductivity vanishes on a subset $\Omega_0 = \Omega \setminus \overline{G}$.
Note that 9-14 can be considered as a problem of mixed type. In the case that μ is constant on Ω and σ is linear and independent of $x \in G$ it is elliptic on Ω_0 (with respect to the space variable x for fixed time) and parabolic on $(0,\infty) \times G$ at least in the case 8. It is shown in this paper that 9-14 admits for given $\mathbf{h}_0$, $\mathbf{j}_0$ and Q_k a unique global weak solution $(\mathbf{E}, \mathbf{h})$ with

$$\mathbf{E} \in W^{-1,\infty}_{loc}\left([0,\infty), L^{r_0}(\Omega \cap B_{R_0}) \cap L^6(\mathbb{R}^3 \setminus B_{R_0})\right) \cap L^{\tilde{r}}_{loc}([0,\infty), \mathcal{G})$$

$$\subset \mathcal{D}'((0,\infty) \times \Omega) \text{ and } \mathbf{h} \in C([0,\infty), L^2(\Omega)),$$

where $\mathcal{G}, r_0 \in (6/5, 2]$ and $\tilde{r}$ as in 6 and $R_0 > 0$, such that $\mathcal{C} \subset B_{R_0} = \{|x| < R_0\}$. Moreover, it is shown that

$$\mathbf{h}_\varepsilon \stackrel{\varepsilon \to 0}{\longrightarrow} \mathbf{h} \text{ in } L^\infty((0,T), L^2(\Omega)) \text{ weak } - * \text{ for all } T \in (0,\infty)$$

$$\text{and } \mathbf{E}_\varepsilon \stackrel{\varepsilon \to 0}{\longrightarrow} \mathbf{E} \text{ in } \mathcal{D}'\left((0,\infty) \times \Omega\right).$$

Finally, the asymptotic behavior of the solution to 9-14 for $t \to \infty$ is investigated in the case $\mathbf{j}_0 = 0$ under a suitable coerciveness-condition on $\mathbf{S}$.
It is shown under certain assumptions on the initial-state $\mathbf{h}_0$ that $||\mathbf{h}(t)||_{L^2}$ decays exponentially.

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[2] Hong-Ming Yin, *Regularity of Solutions to Maxwell's System in Quasi-stationary Electromagnetic Fields and Applications* , Comm. Partial Differential Equations 22 (1997), 1029 - 1053.

[3] F. Jochmann, *A Compactness Result for Vector Fields with divergence and curl in $L^q(\Omega)$ Involving Mixed Boundary Conditions*, Appl. Anal. vol. 66 (1997), pp. 189-203.

[4] A. Milani, R. Picard, *Weak Solution Theory for Maxwell's Equations in the Semistatic Limit*, J. Math. Anal. Appl. 191 (1995), 77 - 100.