Global non-existence and blow up for quasi-linear equations with source and dissipative terms

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The aim of this poster is to illustrate a recent global non–existence result (see [13] for more precise results and proofs) for the solutions of a class of evolution quasi-linear dissipative equations.

The simplest model belonging to the class of equations studied is the hyperbolic problem

$$\begin{cases} u_{tt} - \Delta u + b|u_t|^{m-2}u_t = c|u|^{p-2}u, & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0, & \text{in } (0,T)\times\partial\Omega, \\ u(0,\cdot)\in H_0^1(\Omega), \qquad u_t(0,\cdot)\in L^2(\Omega), \end{cases} \tag{1}$$

where $T>0$, Ω is an open bounded subset of $\mathbb{R}^n$, $n\geq 1$, $u=u(t,x)$, $b,c>0$ e $m>1$. To the solutions of (1) is naturally associated the energy function

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{c}{p}\|u\|_p^p.$$

In the case $b=0$ it is well known (see [1], [3], [6], [7], [9], [12]) that, if $E(0)<0$, the source term $c|u|^{p-2}u$ causes global nonexistence of the solutions, and finite time blow–up of them in some cases. In particular Payne and Sattinger proved in [9] that, if $p<2n/(n-2)$, the non–existence in all $[0,\infty)$ of the solutions (which, by the results of [1], implies finite time blow–up) can be proved also if $\|u_0\|>\lambda_1$, $E(0)<E_1$, for some $\lambda_1, E_1>0$, while if $\|u_0\|_p<\lambda_1$ and $E(0)<E_1$ the solutions are global and bounded in the H_0^1 norm. If $c=0$, $m>2$, the nonlinear damping term $b|u_t|^{m-2}u_t$ assures global existence of the solutions for arbitrary initial data (see [4] and [5]).

In [2] Georgiev and Todorova proved that, if

$$2<m<p\leq 2(n-1)/(n-2) \tag{2}$$

and the initial energy $E(0)$ is sufficiently large and negative, then the solutions of (1) blow up in a finite time. This result has been extended to abstract evolution equations by Levine and Serrin (see [8]) who, moreover, proved that, if $1<m<p$, $p>2$ and $E(0)<0$ then the solutions of (1) cannot exist in the whole $(0,\infty)$. More recently Pucci and Serrin (see [10]) extended the result to cover the case in which $E(0)<E_1$ and $\|u(0)\|_p>\lambda_1$, but $m=2$.

On the other hand, as a consequence of the theorems of Pucci and Serrin in [11], there is $0<E_0<E_1$ such that if $E(0)<E_0$ and $\|u(0)\|_p<\lambda_1$, then the solutions of (1) are asymptotically stable. By combining the ideas of [8] and [11] we prove that if

$$1<m<p, \quad 2<p\leq 2n/(n-2), \tag{3}$$

$E(0)\leq E_1$ and $\|u(0)\|_p>\lambda_1$, the solutions of (1) cannot exist in the whole $(0,\infty)$. Then, by using also the local existence and continuation theorem of [2] we obtain that, if (3) is replaced by the

stronger condition (2), the solutions of (1) blow up in a finite time. In this way we extend the result of Levine –Serrin concerning the case in which $E(0) < 0$ (what implies that $\|u(0)\|_p > \lambda_1$) to the case in which $E(0) \leq E_1$ and $\|u(0)\|_p > \lambda_1$; the last condition excludes the stability case studied in [11].

Moreover in this way we exactly extend the result of Payne and Sattinger concerning the non–dissipative case in which $b = 0$ to the dissipative one, and we extend the result of Pucci and Serrin in [10] to the general case $m > 1$.

We generalize this result by replacing the specific nonlinearities $b|u_t|^{m-2}u_t$ and $c|u|^{p-2}u$ with general repulsive forces $f = f(x, u)$ and (even non–autonomous) damping terms $Q = Q(t, u_t)$ under specific assumptions.

Finally we generalize the result to abstract evolution problems. Some important cases to which our abstract results are applicable are:

(a) the equation of motion of a damped clamped plate under the effects of a given loading function, that is

$$\begin{cases} 2\rho(x)u_{tt} + (-\Delta)^2 u + Q(t, x, u_t) = f(x, u), & \text{in } (0, T) \times \Omega, \\ u(t, x) = 0, Du(t, x) = 0, & \text{in } (0, T) \times \partial\Omega, \\ u(0, \cdot) \in H_0^2(\Omega), \qquad u_t(0, \cdot) \in L^2(\Omega), \end{cases} \tag{4}$$

where Ω and u are as before, and $\rho \in L^\infty(\Omega; \mathbb{R}_0^+)$ represents the density, $d > 0$ the flexural rigidity and $2h > 0$ the thickness of the plate;

(b) the effect of nonlinear forcing and damping terms added to the classical equations of linear elasticity, that is

$$\begin{cases} \rho(x)u_{tt} - \operatorname{div}(C(x)\nabla u) + Q(t, x, u_t) = f(x, u), & \text{in } (0, T) \times \Omega, \\ u(t, x) = 0, & \text{in } (0, T) \times \partial\Omega, \\ u(0, \cdot) \in [H_0^1(\Omega)]^n, \qquad u_t(0, \cdot) \in [L^2(\Omega)]^n, \end{cases} \tag{5}$$

where Ω is as before, $u = u(t, x)$, $u : J \times \Omega \to \mathbb{R}^n$, $n \geq 1$, $\rho \in L^\infty(\Omega; \mathbb{R}_0^+)$ and $C(x)$ is a linear operator from $\mathbb{R}^{2n}$ to itself for all $x \in \Omega$, i.e. a tensor of rank 4 which can be represented as $C(x) = (c_{ijkl}(x))$, $i, j, k, l = 1, \ldots, n$, and $c_{ijkl} \in L^\infty(\Omega)$;

(c) evolutions systems involving the q–Laplacian operator, with $q > 1$, instead of the Laplacian operator.

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