NON-EQUIVALENCE OF THE CONVENTIONAL BOUNDARY INTEGRAL FORMULATION AND ITS ELIMINATION FOR TWO-DIMENSIONAL MIXED POTENTIAL PROBLEMS

Wen-jun He,† Hao-jiang Ding and Hai-chang Hu
Department of Mechanics, Zhejiang University, Hangzhou 310027, People's Republic of China

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Abstract—For a given mixed type potential problem, the corresponding conventional boundary integral equation is shown to yield non-equivalent solutions. Numerical results show that the conventional boundary integral formulation yields incorrect potential and flux results when the distance scale in the fundamental solution approaches its degenerate value. Such a kind of non-equivalence of the conventional boundary integral equation can be eliminated by the use of the necessary and sufficient boundary integral formulation which always ensures the equivalence of solutions. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

The boundary element method (BEM) has emerged as a powerful alternative in the analysis of potential problems. However, as some researchers noticed [1–3], the conventional boundary integral equation (CBIE) may give non-equivalent solutions for plane potential problems or plane harmonic functions. Obviously, a solution of boundary value problem (BVP) is a solution of its corresponding CBIE, but the converse argument may be not true, i.e. a CBIE may not be equivalent to the BVP.

The CBIE formulation for potential problems can be written out as

\[ ku(x) = u_0(\xi)G(x, \xi) - \frac{\partial G(x, \xi)}{\partial n}(\xi) \int dB(\xi), \]

with

\[ G(x, \xi) = -\frac{1}{2\pi} \ln \frac{\rho(x, \xi)}{a}, \]

and

\[ \rho(x, \xi) = \sqrt{(x - \xi)^2 + (y - \eta)^2} \]

where \( B \) represents the boundary composed of the field point \((\xi, \eta)\). The fundamental solution \( G(x, \xi) \) contains an arbitrary distance scale \( a \) introduced in order that the argument of the transcendental function is non-dimensional. The distance scale \( a \) should have no effects on computation to show the objective-ness of the integral equation.

Christiansen [3] proved that the non-equivalence existed for a special Dirichlet problem as follows (Fig. 1a):

\[ \Delta u = 0 \text{ in } \Omega, \]

\[ u = u_1 = \text{const. on } B_1, \]

\[ u = u_0 = \text{const. on } B_0. \]

Obviously, the boundary value problem (BVP) defined by eqn (4) always has a unique solution [1]. However, the related CBIE sometimes may yield a non-unique solution when \( \alpha_0/a = 1 \) [3]. In this case, the boundary integral equation (BIE) is known as non-equivalent to its corresponding BVP. Here the word non-equivalent not non-unique is chosen considering that non-unique solutions may exist for Neumann problems in which a BIE should also have the same number of non-unique solutions. It was found that the exceptional case depends on the distance scale in the fundamental solution and the exterior boundary \( B_0 \). For a given exterior boundary \( B_0 \), there is a special magnitude of the distance scale to make BIE solution non-equivalent. This value of the distance scale can be named as degenerate scale.

Christiansen [3, 4] also derived a supplementary condition of the integral equation for plane potential problems by virtue of Green's third identity. When the supplementary condition is used, a unique solution can always be obtained. However, an over-determined system with more equations than unknowns is obtained and the least-square solution satisfies the supplementary condition in an approximate fashion.
only. Furthermore, only Dirichlet problems were investigated in his paper.

Another method to avoid the difficulties due to the non-equivalence recommended by Muskhehshvili [1] and other researchers is simply changing the scale value for a given geometry so that it is away from its degenerate value. However, such a degenerate value is generally unknown for most cases. On the other hand, it would be easy to keep a secure distance from the critical value simply by introducing a distance scale into the fundamental solution in advance if this degenerate scale value is known beforehand. Hence, the scaling method is not feasible especially that under most practical circumstances even a correct solution is unknown.

The basic reason of non-equivalence of CBIE solution was that the simple layer formula of space harmonic functions was transferred improperly to be used for plane harmonic functions [5]. Indeed, the simple layer formula of plane harmonic functions cannot cover every harmonic function.

A complete way to eliminate the non-equivalence is to use a necessary and sufficient BIE (NSBIE). A BIE formulation is called as necessary and sufficient when a BIE solution is also a solution of the BVP and vice versa. Such an NSBIE formulation has been established by Hu [5], Telles and De Paula [6] for potential problems:

\[
ku(x) = \int_{\partial B} \left( u_\ast(\xi)G(x, \xi) - u(\xi) \frac{\partial G(x, \xi)}{\partial n} \right) dB(\xi) + \alpha,
\]

where \( \alpha \) is a new unknown constant and eqn (6) is a new equation. \( k \) in eqns (1) and (5) is known and equals to \( \frac{1}{\rho} \) on a smooth boundary. Compared with CBIE, the NSBIE formulation possesses objectiveness and its form will not change due to the distance scale variation. Though the non-objectiveness of the conventional equation can be used to avoid difficulties of non-unique solutions, for instance, the scaling method, there are some shortcomings which are hard to overcome as indicated before.

One hazardous phenomenon of using CBIE is that its non-equivalence of solution will be transferred into a unique solution after discretization of the equation in numerical analysis. This may lead to a fictitious solution which, in general, is hard to be realized. However, NSBIE can eradicate this possibility completely.

Christiansen's work is significant because it was the first study where a special example was presented and quantified errors of CBIE were able to be exhibited so that CBIE was known to have such problems. However, only those Dirichlet type examples were presented. On the other hand, most practical problems are of mixed type, and not Dirichlet. Therefore, it is significant to investigate the not fully understood CBIE behaviour for mixed boundary value problems, i.e. studying mixed problems and considering their meaningful practical background. One objective of this paper is to prove that CBIE may yield non-equivalent solutions in comparison with its corresponding BVP for a given mixed potential problem.

Another objective is to investigate the numerical behaviour of CBIE and NSBIE for mixed problems. It is important to know how the non-equivalence of CBIE will demonstrate in numerical analysis.

2. NON-EQUIVALENCE OF CBIE SOLUTIONS FOR MIXED POTENTIAL PROBLEMS

The following mixed problem in an annular region composed of two concentric circles (Fig. 1b) is to be studied here.

\[
\begin{align*}
\Delta u &= 0 \text{ in } \Omega, \\
u &= u_1 = \text{const. on } B_1 : r = r_1, \\
u &= u_m = \text{const. on } B_2 : r = r_0.
\end{align*}
\]

(7)

CBIE [eqn (1)] can be shown to yield non-equivalent solutions for such a problem when \( r_0/\rho = 1 \).

Rearrange eqn (1) by putting all known variables on the right-hand side:

\[
ku(x) - \int_{\partial B_2} u_\ast(\xi)G(x, \xi)dB(\xi) = \alpha - \int_{\partial B_1} u_\ast(\xi)dB(\xi),
\]

(5)

where \( \alpha \) is a new unknown constant and eqn (6) is a new equation. \( k \) in eqns (1) and (5) is known and equals to \( \frac{1}{\rho} \) on a smooth boundary. Compared with CBIE, the NSBIE formulation possesses objectiveness and its form will not change due to the distance scale variation. Though the non-objectiveness of the conventional equation can be used to avoid difficulties of non-unique solutions, for instance, the scaling method, there are some shortcomings which are hard to overcome as indicated before.

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Conventional boundary integral formulation

Table 1. Constant element

<table>
<thead>
<tr>
<th>N</th>
<th>r₀/a</th>
<th>u₀</th>
<th>u₁</th>
<th>u₀</th>
<th>u₁</th>
<th>Analytical</th>
</tr>
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<td>-0.52</td>
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Table 2. Linear element

<table>
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<th>u₀</th>
<th>u₁</th>
<th>u₀</th>
<th>u₁</th>
<th>Analytical</th>
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<td>144.27</td>
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</table>

The determinant about u₀ and u₁ is

\[ \Delta = r₁ \ln \frac{r₀}{a} \]

Obviously u₀ and u₁ can have arbitrary values when \( r₀/a = 1 \) no matter what value of \( r₁ \) is if \( u₁ ≠ 0 \). It should be emphasized that eqn (6) may not be satisfied automatically and \( a \) in eqn (5) will not be zero at this time which will be demonstrated later in numerical analysis. That the homogeneous eqns (10) and (11) have non-trivial solutions means that its corresponding non-homogeneous equation eqn (8) must have non-unique solutions. However, the mixed problem of eqn (7) can only have a unique solution. Thus, the non-equivalence of solution of CBIE is proved and this non-equivalent solution does not depend on the internal radius \( r₁ \).

Coincidentally, it is interesting that the degenerate scale for this mixed problem is the same as for the Dirichlet problem, eqn (4). Therefore, the concept of transfinite diameter [7] may be used to represent the exceptional case here for a mixed problem as for a Dirichlet problem. As the same as in Dirichlet problem, there is an eigenvalue of the integral equation equal to zero if the transfinite diameter of the considered boundary curve is equal to the distance scale. For a doubly connected region, the transfinite diameter of the exterior boundary curve is crucial.

For a circle with radius \( r \):

\[ d = r. \]

For a ellipse with semi-axes \( a₁ \) and \( b₁ \):

\[ d = \frac{a₁ + b₁}{2}. \]

3. NUMERICAL EXAMPLES

Computations are tested to CBIE and NSBIE respectively for the following examples.

Example 1

This example is to solve Laplace equation with mixed boundary condition as follows in a region composed of two concentric circles with radii \( r₀ \) and \( r₁ \), respectively (Fig. 1b).

The boundary condition is

\[ u₀ = \frac{100.0}{r₁ \ln \frac{r₁}{r₀}}, \quad u₁ = 100.0. \]

The unique analytical solution is [8]

\[ u = u₀ + \frac{r₁}{r₀} \ln \frac{r₁}{r₀} \]

\[ u₁ = u₀ + \frac{r₁}{r₀} (u₁ - u₀), \quad r₁ ≤ r ≤ r₀. \]
Set $r_0 = 2r_1$ and place the same amount of elements on internal and exterior circles. Computations are carried out about three different sizes and element numbers for three kinds of element: constant, linear and quadratic. The results are listed in Tables 1–3, where $N$ denotes total element number. From Tables 1–3, we find that at the degenerate scale value ($r_0/a = 1$), all CBIE potential and flux results are wrong, but NSBIE yields accurate results no matter which element and how many elements are used.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$r_0/a$</th>
<th>$u_0$</th>
<th>$u_{n1}$</th>
<th>$u_0$</th>
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<td>−0.31</td>
<td>144.71</td>
<td>0.0031</td>
<td>144.27</td>
</tr>
</tbody>
</table>

Fig. 2. Potential on the exterior circle $u_0$: (a) constant elements; (b) linear elements; (c) quadratic elements.
Example 2

To investigate the CBIE and NSBIE behaviour in the vicinity of the degenerate scale, considered here is the same case with Example 1 except $0.5 \leq r_0/a \leq 1.5$. The results of $u_0$ and $u_{ai}$ are depicted in Figs 2 and 3 for constant, linear and quadratic elements, respectively.

In addition to the same conclusions as in Example 1, the following points can also be found from Figs 2 and 3:

(1) The CBIE results are incorrect when $r_0/a$ approaches the degenerate scale no matter how fine

Fig. 3. Flux on the interior circle $u_{ai}$: (a) constant elements; (b) linear elements; (c) quadratic elements.

Fig. 4. Mixed problem in a double connected domain composed of two confocal ellipses.
Table 4. Linear element

<table>
<thead>
<tr>
<th>N</th>
<th>c/a</th>
<th>u_0(B)</th>
<th>u_0(A)</th>
<th>u_0(B)</th>
<th>u_0(A)</th>
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<td>2917.2</td>
<td>50.437</td>
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Table 5. Quadratic element

<table>
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<th>u_0(B)</th>
<th>u_0(A)</th>
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<td>50.437</td>
<td>156.68</td>
<td>50.916</td>
</tr>
</tbody>
</table>

meshes for constant and linear elements. The NSBIE results are always stable and accurate. CBIE's instability is also shown by the fact that the CBIE results using constant elements have different trends compared with those using linear and quadratic elements.

(2) For quadratic element results, the above conclusion can be derived also if the element mesh is not so dense (N = 16). If the mesh is fine enough (N = 48), it is interesting to notice that the CBIE results are also relatively accurate near the degenerate scale. It may be interpreted by the concept of degenerate interval which denotes the range of inaccurate CBIE results. When enough quadratic elements (N = 48) are utilized, the degenerate interval is relatively small. This implies CBIE errors around the degenerate scale can be reduced to a certain degree when a lot of high order elements are employed. However, the advantage of boundary elements over finite elements seems to be lost in this case.

Example 3

This is a mixed problem defined in a domain composed of two confocal ellipses (Fig. 4). This problem, as a classical example to investigate the accuracy of BEM, was studied in Refs [3] and [9]. The boundary curves are:

\[ B_0: x = a_0 \cos \theta, \quad y = b_0 \sin \theta; \quad 0 \leq \theta \leq 2\pi, \]

\[ B_1: x = a_1 \cos \theta, \quad y = b_1 \sin \theta; \quad 0 \leq \theta \leq 2\pi, \]

\[ a_0 = c \cos h \mu_0, \quad b_0 = c \sin h \mu_0, \]

\[ a_1 = c \cos h \mu_1, \quad b_1 = c \sin h \mu_1, \]

where \( 0 < \mu_1 < \mu_0 \), and \( c \) is a constant. The problem has a unique analytical solution:

\[ u = u_0 + \frac{\mu_0 - \mu}{\mu_0 - \mu_1} (u_1 - u_0), \quad \mu_1 \leq \mu \leq \mu_0. \]  

4. CONCLUSIONS

CBIE has non-equivalence of solutions when the distance scale is improperly chosen for a given problem. After boundary element discretization, the linear system of equations also has a degenerate value with respect to the distance scale which is not always the same as the accurate CBIE degenerate scale values before the boundary element approximation. Also, the denser the mesh, the nearer are these two degenerate values.

When the distance scale approaches its degenerate scale value, the CBIE accuracy is greatly affected. The magnitude of the degenerate interval in which solution accuracy has been damaged greatly decreases with the refinement of the boundary element mesh.

NSBIE eliminates the difficulties due to non-equivalence of solutions and thus is able to give correct and stable solutions. The computing work will only increase a little, if at all, and it is convenient to incorporate NSBIE formulation into a boundary element program.

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REFERENCES


