

# MAXIMAL $L^p$ -REGULARITY FOR STOCHASTIC EVOLUTION EQUATIONS

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ABSTRACT. We prove maximal  $L^p$ -regularity for the stochastic evolution equation

$$\begin{cases} dU(t) + AU(t) dt = F(t, U(t)) dt + B(t, U(t)) dW_H(t), & t \in [0, T], \\ U(0) = u_0, \end{cases}$$

under the assumption that  $A$  is a sectorial operator with a bounded  $H^\infty$ -calculus of angle less than  $\frac{1}{2}\pi$  on a space  $L^q(\mathcal{O}, \mu)$ . The driving process  $W_H$  is a cylindrical Brownian motion in an abstract Hilbert space  $H$ . For  $p \in (2, \infty)$  and  $q \in [2, \infty)$  and initial conditions  $u_0$  in the real interpolation space  $D_A(1 - \frac{1}{p}, p)$  we prove existence of unique strong solution with trajectories in

$$L^p(0, T; D(A)) \cap C([0, T]; D_A(1 - \frac{1}{p}, p)),$$

provided the non-linearities  $F : [0, T] \times D(A) \rightarrow L^q(\mathcal{O}, \mu)$  and  $B : [0, T] \times D(A) \rightarrow \gamma(H, D(A^{\frac{1}{2}}))$  are of linear growth and Lipschitz continuous in their second variables with small enough Lipschitz constants. Extensions to the case where  $A$  is an adapted operator-valued process are considered as well.

Various applications to stochastic partial differential equations are worked out in detail. These include higher-order and time-dependent parabolic equations and the Navier-Stokes equation on a smooth bounded domain  $\mathcal{O} \subseteq \mathbb{R}^d$  with  $d \geq 2$ . For the latter, the existence of a unique strong local solution with values in  $(H^{1,q}(\mathcal{O}))^d$  is shown.

## 1. INTRODUCTION

Maximal  $L^p$ -regularity techniques have been pivotal in much of the recent progress in the theory of parabolic evolution equations (see [2, 22, 25, 54, 76, 86] and there references therein). Among other things, such techniques provide a systematic and powerful tool to study nonlinear and time-dependent parabolic problems.

For stochastic parabolic evolution equations, maximal  $L^p$ -regularity results have been obtained previously by Krylov for second order problems on  $\mathbb{R}^d$  [44, 46, 47, 48, 49], by Kim for second order problems on bounded domains in  $\mathbb{R}^d$  [43], and by Mikulevicius and Rozovskii for Navier-Stokes equations [63]. A systematic theory of maximal  $L^p$ -regularity for stochastic evolution equations, however, based on

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abstract operator-theoretic properties of the operators governing the equation, has yet to be developed. A first step towards such a theory has been taken in our recent paper [68], where it was shown that if  $A$  is a sectorial operator with a bounded  $H^\infty$ -calculus of angle  $< \frac{1}{2}\pi$  on a space  $L^q(\mathcal{O}, \mu)$  with  $(\mathcal{O}, \mu)$  an arbitrary  $\sigma$ -finite measure space and  $q \in [2, \infty)$ , then  $A$  has stochastic maximal  $L^p$ -regularity for all  $p \in (2, \infty)$ , i.e.,  $A$  satisfies the convolution estimate

$$(1.1) \quad \left\| t \mapsto \int_0^t A^{\frac{1}{2}} S(t-s) G(s) dW_H(s) \right\|_{L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}, \mu))} \leq C \|G\|_{L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}, \mu; H))},$$

where  $S$  denotes the semigroup generated by  $-A$  and  $W_H$  is a cylindrical Brownian motion in a Hilbert space  $H$ . The stochastic integral is understood as a vector-valued stochastic integral in  $L^q(\mathcal{O}, \mu)$  in the sense of [65].

The aim of this paper is to apply the above estimate to deduce maximal  $L^p$ -regularity for the stochastic parabolic evolution equation

$$\begin{cases} dU(t) + AU(t) dt = F(t, U(t)) dt + B(t, U(t)) dW_H(t), & t \in [0, T], \\ U(0) = u_0, \end{cases}$$

Our main result asserts that if  $A$  has a bounded  $H^\infty$ -calculus of angle  $< \frac{1}{2}\pi$  on a Banach space  $X$  that is isomorphic to a closed subspace of  $L^q(\mathcal{O}, \mu)$  with  $q \in [2, \infty)$ , then for  $p \in (2, \infty)$  and initial conditions  $u_0$  in the real interpolation space  $\mathbb{D}_A(1 - \frac{1}{p}, p) = (X, \mathbb{D}(A))_{1 - \frac{1}{p}, p}$ , this problem has a unique strong solution with trajectories in

$$L^p(0, T; \mathbb{D}(A)) \cap C([0, T]; \mathbb{D}_A(1 - \frac{1}{p}, p)),$$

provided the non-linearities  $F : [0, T] \times \mathbb{D}(A) \rightarrow X$  and  $B : [0, T] \times \mathbb{D}(A) \rightarrow \gamma(H, \mathbb{D}(A^{\frac{1}{2}}))$  are of linear growth and Lipschitz continuous in their second variables with small enough Lipschitz constants. The precise statement is contained in Theorem 4.5, where we allow  $A$ ,  $F$  and  $B$ ,  $u_0$  to be random.

To illustrate the power of this result, we apply it to the time-dependent problem

$$\begin{cases} dU(t) + A(t)U(t) dt = F(t, U(t)) dt + B(t, U(t)) dW_H(t), & t \in [0, T], \\ U(0) = u_0, \end{cases}$$

and show in Theorem 5.2 that, essentially under the same assumptions as in the time-independent case, the same conclusions can be drawn with regard to the existence, uniqueness, and regularity of strong solutions. An extension to the case of locally Lipschitz continuous coefficients is given in Subsection 5.2. These results extend [7, Theorems 4.3 and 4.10], [89, Theorem 2.5] and [91, Theorem 6.1] to the case of sharp exponents.

It has already been mentioned that in Theorem 4.5 we allow  $A$  to be random. In the special case where  $A$  is a fixed deterministic operator, the theorem can be applied (by taking the negative extrapolation space  $\mathbb{D}(A^{-\frac{1}{2}})$  as the state space) to the situation where the non-linearities are of the form  $F : [0, T] \times \mathbb{D}(A^{\frac{1}{2}}) \rightarrow \mathbb{D}(A^{-\frac{1}{2}})$  and  $B : [0, T] \times \mathbb{D}(A^{\frac{1}{2}}) \rightarrow \gamma(H, X)$ . For initial values in  $\mathbb{D}_A(\frac{1}{2} - \frac{1}{p}, p)$ , this results in solutions with trajectories in  $L^p(0, T; \mathbb{D}(A^{\frac{1}{2}})) \cap C([0, T]; \mathbb{D}_A(\frac{1}{2} - \frac{1}{p}, p))$ . For second order elliptic operators  $A$  on a smooth domain  $\mathcal{O} \subseteq \mathbb{R}^d$ , this includes the case where  $F$  and  $B$  arise as Nemytskii operators associated with nonlinear functions of the form  $f(u, \nabla u)$  and  $b(u, \nabla(u))$ . This is because in this setting  $\mathbb{D}(A^{\frac{1}{2}})$  typically can

be identified as a Sobolev space  $H^{1,q}$ . An illustration is given in Section 8, where we prove existence of solutions in  $H^{1,q}$  for the stochastic Navier-Stokes equation.

The advantage of the abstract approach presented in this paper is that it replaces some of the hard (S)PDE techniques of Krylov's  $L^p$ -theory by the generic assumption that  $A$  have a good functional calculus. In recent years, a large body of results has been accumulated by many authors which shows that, as a rule of thumb, any 'reasonable' elliptic operator of order  $2m$  has such a calculus (see [3, 21, 22, 26, 27, 28, 29, 40, 41, 54, 59, 71, 87] and the references therein); much of the hard analysis goes into proving these ready-to-use results. Moreover, in most of these examples, the trace space  $D_A(1 - \frac{1}{p}, p)$  and the fractional domain space  $D(A^{\frac{1}{2}})$  have been characterised explicitly as a fractional Besov space of order  $2m(1 - \frac{1}{p})$  and a Sobolev space of order  $m$ , respectively.

**1.1. Applications.** In principle, our results pave the way for proving maximal  $L^p$ -regularity results for any parabolic problem governed by an operator having a bounded  $H^\infty$ -calculus.

To keep this paper at a reasonable length we have picked three examples which we believe to be representative (but by no means exhaustive) to illustrate the scope of applications. Further potential applications include, for instance, parabolic SPDEs on complete Riemannian manifolds and on Wiener spaces such as considered in [90] (cf. Examples 3.2 (7) and (8) below).

**1.1.1. Higher-order parabolic SPDEs on  $\mathbb{R}^d$ .** Our first application concerns a system of  $N$  coupled parabolic SPDEs involving elliptic operators of order  $2m$  on  $\mathbb{R}^d$  of the form

$$\begin{cases} du(t, x) + \mathcal{A}(t, x, D)u(t, x) dt = f(t, x, u) dt + \sum_{i \geq 1} b_i(t, x, u) dw_i(t), \\ u(0, x) = u_0(x). \end{cases}$$

Here

$$\mathcal{A}(t, \omega, x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(t, \omega, x) D^\alpha$$

with  $D = -i(\partial_1, \dots, \partial_d)$ . The scalar Brownian motions  $w_i$  are independent, and the functions  $f$  and  $b_i$  are Lipschitz continuous with respect to the graph norm of  $\mathcal{A}$ . Under suitable boundedness and continuity assumptions on the coefficients  $a_\alpha$  and a smallness condition on Lipschitz constants of  $f$  and  $b_i$  we prove the existence and uniqueness of a strong solution with values in  $H^{2m,q}(\mathbb{R}^d; \mathbb{C}^N)$  and with continuous trajectories in the Besov space  $B_{q,p}^{2m(1-\frac{1}{p})}(\mathbb{R}^d; \mathbb{C}^N)$  (Theorem 6.3). To the best of our knowledge, this is the first maximal  $L^p$ -regularity result for this class of equations.

**1.1.2. Time-dependent second-order parabolic SPDEs on bounded domains.** As a second example we consider time-dependent parabolic second order problems on a bounded domain  $\mathcal{O} \subseteq \mathbb{R}^d$  whose boundary consists of two disjoint arcs  $\partial\mathcal{O} = \Gamma_0 \cup \Gamma_1$ . We impose Dirichlet conditions on  $\Gamma_0$  and Neumann conditions on  $\Gamma_1$  and prove the existence of a unique strong solution with values in  $H^{2,q}(\mathcal{O})$  and with continuous trajectories in the Besov space  $B_{q,p}^{2-\frac{2}{p}}(\mathcal{O})$  (Theorem 7.3).

1.1.3. *The Navier-Stokes equation on bounded domains.* In the final section we consider the stochastic Navier-Stokes equation in a bounded smooth domain  $\mathcal{O} \subseteq \mathbb{R}^d$  with  $d \geq 2$  subject to Dirichlet boundary conditions. We prove existence and uniqueness of a local mild solution with values in  $(H^{1,q}(\mathcal{O}))^d$  and with continuous trajectories in  $(B_{q,p}^{1-\frac{2}{p}}(\mathcal{O}))^d$  for  $\frac{d}{2q} < 1 - \frac{2}{p}$ .

## 2. PRELIMINARIES

The aim of this section is to fix notations and to recall some recent results on maximal  $L^p$ -regularity and stochastic maximal  $L^p$ -regularity that will be needed in the sequel.

Throughout this article we fix a probability spaces  $(\Omega, \mathcal{A}, \mathbb{P})$  endowed with filtration  $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ , a Hilbert space  $H$  with inner product  $[\cdot, \cdot]$ , and a Banach space  $X$ .

For  $p_1, p_2 \in [1, \infty]$ , the closed linear span in  $L^{p_1}(\Omega; L^{p_2}(\mathbb{R}_+; X))$  of all processes of the form  $f = \mathbf{1}_{(s,t] \times F} \otimes x$  with  $F \in \mathcal{F}_s$  and  $x \in X$  is denoted by

$$L_{\mathcal{F}}^{p_1}(\Omega; L^{p_2}(\mathbb{R}_+; X)).$$

The elements in  $L_{\mathcal{F}}^{p_1}(\Omega; L^{p_2}(\mathbb{R}_+; X))$  will be referred to as the  $\mathcal{F}$ -adapted elements in  $L^{p_1}(\Omega; L^{p_2}(\mathbb{R}_+; X))$ .

The vector space of all (equivalence classes of) strongly measurable functions on  $\Omega$  with values in a Banach space  $Y$  is denoted by  $L^0(\Omega; Y)$ . The topology of convergence in probability is metrised by the distance function  $d(f, g) = \mathbb{E}(\|f - g\| \wedge 1)$  which turns  $L^0(\Omega; Y)$  into a complete metric vector space. The space of all  $f \in L^0(\Omega; Y)$  that are strongly  $\mathcal{B}$ -measurable, where  $\mathcal{B} \subseteq \mathcal{A}$  is a sub- $\sigma$ -algebra, is denoted by  $L_{\mathcal{B}}^0(\Omega; Y)$ .

**2.1. Stochastic integration.** We will be interested in an estimate for stochastic integrals of the form  $\int_{\mathbb{R}_+} G dW_H$ , where  $G$  is an  $\mathcal{F}$ -adapted process with values in space of finite rank operators from  $H$  to  $X$ , and  $W_H$  is an  $\mathcal{F}$ -cylindrical Brownian motion in  $H$ . We start with a concise explanation of these notions.

2.1.1. *The space  $\gamma(\mathcal{H}, X)$ .* Let  $\mathcal{H}$  be a Hilbert spaces (typically we take  $\mathcal{H} = H$  or  $\mathcal{H} = L^2(\mathbb{R}_+; H)$ ). The space of all  $\gamma$ -radonifying operators from  $\mathcal{H}$  to  $X$  is denoted by  $\gamma(\mathcal{H}, X)$ . Recall that this space is the closure of the space of finite rank operators from  $\mathcal{H}$  to  $X$  with respect to the norm

$$\left\| \sum_{n=1}^N h_n \otimes x_n \right\|_{\gamma(\mathcal{H}, X)}^2 := \mathbb{E} \left\| \sum_{n=1}^N \gamma_n \otimes x_n \right\|^2,$$

where it is assumed that  $(h_n)_{n=1}^N$  is an orthonormal sequence in  $\mathcal{H}$ ,  $(x_n)_{n=1}^N$  is a sequence in  $X$ , and  $(\gamma_n)_{n=1}^N$  is any sequence of independent standard Gaussian random variables. For expositions of the theory of  $\gamma$ -radonifying operators we refer to [24] and the review article [64], where also references to the extensive literature can be found.

For  $X = L^p(\mathcal{O}, \mu)$  with  $1 \leq p, \infty$  and  $(\mathcal{O}, \mu)$   $\sigma$ -finite, one has a canonical isomorphism

$$(2.1) \quad L^p(\mathcal{O}, \mu; \mathcal{H}) \simeq \gamma(\mathcal{H}, L^p(\mathcal{O}, \mu))$$

which is obtained by assigning to a function  $f \in L^p(\mathcal{O}, \mu; \mathcal{H})$  the operator  $T_f : H \rightarrow L^p(\mathcal{O}, \mu)$ ,  $h \mapsto [f(\cdot), h]$  (see [10]). More generally the same procedure gives, for any Banach space  $X$ , a canonical isomorphism

$$(2.2) \quad L^p(\mathcal{O}, \mu; \gamma(\mathcal{H}, X)) \simeq \gamma(\mathcal{H}, L^p(\mathcal{O}, \mu; X))$$

(see [65]). We shall need the following variation on this theme. Recalling the definition of the Bessel potential spaces  $H^{2\alpha, p}(\mathcal{O})$ , where  $\mathcal{O} \subseteq \mathbb{R}^n$  is a smooth domain, application of the operator  $(I - \Delta)^{-\alpha}$  on both sides of (2.2) gives an isomorphism

$$(2.3) \quad H^{2\alpha p}(\mathcal{O}; \gamma(\mathcal{H}, X)) \simeq \gamma(\mathcal{H}, H^{2\alpha, p}(\mathcal{O}; X)).$$

**2.1.2. Cylindrical Brownian motions.** An  $\mathcal{F}$ -cylindrical Brownian motion in  $H$  is a bounded linear operator  $W_H : L^2(\mathbb{R}_+; H) \rightarrow L^2(\Omega)$  such that:

- (i) for all  $f \in L^2(\mathbb{R}_+; H)$  the random variable  $W_H(f)$  is centred Gaussian.
- (ii) for all  $t \in \mathbb{R}_+$  and  $f \in L^2(\mathbb{R}_+; H)$  with support in  $[0, t]$ ,  $W_H(f)$  is  $\mathcal{F}_t$ -measurable.
- (iii) for all  $t \in \mathbb{R}_+$  and  $f \in L^2(\mathbb{R}_+; H)$  with support in  $[t, \infty)$ ,  $W_H(f)$  is independent of  $\mathcal{F}_t$ .
- (iv) for all  $f_1, f_2 \in L^2(\mathbb{R}_+; H)$  we have  $\mathbb{E}(W_H(f_1) \cdot W_H(f_2)) = [f_1, f_2]_{L^2(\mathbb{R}_+; H)}$ .

It is easy to see that for all  $h \in H$  the process  $(W_H(t)h)_{t \geq 0}$  defined by

$$W_H(t)h := W_H(\mathbf{1}_{(0, t]} \otimes h)$$

is an  $\mathcal{F}$ -Brownian motion (which is standard if  $\|h\| = 1$ ). Moreover, two such Brownian motions  $((W_H(t)h_1)_{t \geq 0})$  and  $((W_H(t)h_2)_{t \geq 0})$  are independent if and only if  $h_1$  and  $h_2$  are orthogonal in  $H$ .

*Example 2.1* (Space-time white noise). Any space-time white noise  $W$  on a domain  $\mathcal{O} \subseteq \mathbb{R}^d$  defines a cylindrical Brownian motion in  $L^2(\mathcal{O})$  and vice versa by the formula

$$W_{L^2(\mathcal{O})}(\mathbf{1}_{(0, t]} \otimes \mathbf{1}_B) = W(t, B)$$

for Borel sets  $B \subseteq \mathcal{O}$  of finite measure.

*Example 2.2* (Sums of independent Brownian motions). A family  $(w_i)_{i \in I}$  of independent real-valued standard Brownian motions defines a cylindrical Brownian motion in  $\ell^2(I)$  and vice versa by

$$W_{\ell^2(I)}(\mathbf{1}_{(0, t]} \otimes e_i) := w_i(t),$$

where  $e_i \in \ell^2(I)$  is given by  $e_i(j) = \delta_{ij}$ .

**2.1.3. The stochastic integral.** Processes which are finite linear combinations of processes of the form

$$\mathbf{1}_{(s, t] \times F} \otimes (h \otimes x)$$

with  $F \in \mathcal{F}_s$ ,  $h \in H$ ,  $x \in X$ , are called  $\mathcal{F}$ -adapted finite rank step processes in  $\gamma(H, X)$ . The stochastic integral of such a process with respect to an  $\mathcal{F}$ -cylindrical Brownian motion  $W_H$  is defined by

$$\int_{\mathbb{R}_+} \mathbf{1}_{(0, t] \times F} \otimes (h \otimes x) dW_H := \mathbf{1}_F [W_H(t)h] \otimes x$$

and linearity. The following two-sided estimate has been proved in [65]:

**Theorem 2.3.** *Let  $X$  be a UMD Banach space and let  $G$  be an  $\mathcal{F}$ -adapted finite rank step process in  $\gamma(H, X)$ . For all  $p \in (1, \infty)$  one has the two-sided estimate*

$$(2.4) \quad \mathbb{E} \left\| \int_{\mathbb{R}_+} G(s) dW_H(s) \right\|^p \approx_p \mathbb{E} \|G\|_{\gamma(L^2(\mathbb{R}_+; H), X)}^p,$$

with implicit constants depending only on  $p$  and (the UMD constant of)  $X$ .

This equivalence is used to give a meaning to the stochastic integral on the left-hand side of the maximal  $L^p$ -regularity inequality (1.1) and plays a crucial role in the proof of this inequality; the inequality (2.5) does not suffice for this purpose (see [68]).

Examples of UMD spaces are all Hilbert spaces and the spaces  $L^q(\mathcal{O}, \mu)$  with  $q \in (1, \infty)$ . Furthermore, closed subspaces, quotients, and duals of UMD spaces are UMD. For more information on UMD spaces we refer to [14].

As a consequence of Theorem 2.3 and a routine density argument, the stochastic integral can be uniquely extended to the space  $L^p_{\mathcal{F}}(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ , which is defined as the closed linear span in  $L^p(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$  of all  $\mathcal{F}$ -adapted finite rank step processes in  $\gamma(H, X)$ . For a detailed discussion we refer to [65].

For Banach space  $X$  with type 2 one has a continuous embedding

$$L^2(\mathbb{R}_+; \gamma(H, X)) \hookrightarrow \gamma(L^2(\mathbb{R}_+; H), X)$$

(see [69, 77]). In combination with (2.4) this gives the following estimate, valid for finite rank step process in  $\gamma(H, X)$  with  $X$  a UMD space with type 2:

$$(2.5) \quad \mathbb{E} \left\| \int_{\mathbb{R}_+} G(s) dW_H(s) \right\|^p \leq C^p \mathbb{E} \|G\|_{L^2(\mathbb{R}_+; \gamma(H, X))}^p.$$

As a consequence of the inequality (2.5), the stochastic integral uniquely extends to  $L^p_{\mathcal{F}}(\Omega; L^2(\mathbb{R}_+; \gamma(H, X)))$ , the closed linear span in  $L^p(\Omega; L^2(\mathbb{R}_+; \gamma(H, X)))$  of all  $\mathcal{F}$ -adapted finite rank step processes in  $\gamma(H, X)$ .

Examples of UMD spaces with type 2 are all Hilbert spaces and the spaces  $L^q(\mathcal{O}, \mu)$  with  $q \in [2, \infty)$ . A UMD space has type 2 if and only if it has martingale type 2, and in fact the estimate (2.5) holds for any Banach space  $X$  with martingale type 2 (see [7, 70]). For more information on the notions of (martingale) type and cotype we refer to [24, 74, 75].

*Remark 2.4.* It follows easily from [55] that the estimates (2.4) and (2.5) are valid for arbitrary exponents  $p \in (0, \infty)$ . We shall not need this fact here.

**2.1.4. The stochastic integral operator family  $\mathcal{J}$ .** We turn our attention to a class of stochastic integral operators, which plays a key role in connection with stochastic maximal  $L^p$ -regularity (see Theorem 3.5 below).

For an  $\mathcal{F}$ -adapted finite rank step process  $G : \mathbb{R}_+ \times \Omega \rightarrow \gamma(H, X)$  and a parameter  $\delta > 0$  we define the process  $J(\delta)G : \mathbb{R}_+ \times \Omega \rightarrow X$  by

$$(J(\delta)G)(t) := \frac{1}{\sqrt{\delta}} \int_{(t-\delta) \vee 0}^t G(s) dW_H(s).$$

A routine computation using (2.5) shows that if  $X$  is a UMD space with type 2 (or, more generally, a Banach space with martingale type 2), then for all  $p \in [2, \infty)$  the mapping  $G \mapsto J(\delta)G$  extends to a bounded operator from  $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$  to  $L^p(\mathbb{R}_+ \times \Omega; X)$  and the family

$$(2.6) \quad \mathcal{J} := \{J(\delta) : \delta > 0\}$$

is uniformly bounded. It what follows, it will be important to know under what additional conditions this family is *R-bounded*.

**2.2. *R*-boundedness.** Let  $X$  and  $Y$  be Banach spaces and let  $(r_n)_{n \geq 1}$  be a Rademacher sequence. A family  $\mathcal{T}$  of bounded linear operators from  $X$  to  $Y$  is called *R-bounded* if there exists a constant  $C \geq 0$  such that for all finite sequences  $(x_n)_{n=1}^N$  in  $X$  and  $(T_n)_{n=1}^N$  in  $\mathcal{T}$  we have

$$\mathbb{E} \left\| \sum_{n=1}^N r_n T_n x_n \right\|^2 \leq C^2 \mathbb{E} \left\| \sum_{n=1}^N r_n x_n \right\|^2.$$

The least admissible constant  $C$  is called the *R-bound* of  $\mathcal{T}$ , notation  $R(\mathcal{T})$ . For Hilbert spaces  $X$  and  $Y$ , *R*-boundedness is equivalent to uniform boundedness and  $R(\mathcal{T}) = \sup_{t \in \mathcal{T}} \|T\|$ . The notion of *R*-boundedness has played an important role in recent progress in the regularity theory of (deterministic) parabolic evolution equations (see Theorem 3.3 below). For more information on *R*-boundedness and its applications we refer the reader to [18, 22, 54].

In Theorems 3.5, 4.5, 5.2, and 5.6 it will be important to have conditions under which the operator family  $\mathcal{J}$  introduced in (2.6) is not just uniformly bounded, but even *R*-bounded, from  $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$  to  $L^p(\mathbb{R}_+ \times \Omega; X)$ . Whether or not this happens depends on the choice of  $p$  and the geometry of the Banach space  $X$ . The proof of next proposition ([68, Theorem 3.1]) depends critically upon the two-sided estimate provided by Theorem 2.3.

**Theorem 2.5** (Conditions for *R*-boundedness of  $\mathcal{J}$ ). *In each of the two cases below,  $\mathcal{J}$  is *R*-bounded as a family of operators from  $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$  to  $L^p(\mathbb{R}_+ \times \Omega; X)$ :*

- (1)  $p \in [2, \infty)$  and  $X$  is isomorphic to a Hilbert space.
- (2)  $p \in (2, \infty)$  and  $X$  is isomorphic to a closed subspace of  $L^q(\mathcal{O}, \mu)$ , with  $q \in (2, \infty)$  and  $(\mathcal{O}, \mu)$  a  $\sigma$ -finite measure space.

The proof of this theorem generalises to 2-convex UMD Banach lattices  $X$  with type 2 over  $(\mathcal{O}, \mu)$  whose 2-concavification  $X_{(2)}$  (see [57, Section 1.d]) is a UMD Banach lattice as well. Further results about the *R*-boundedness of  $\mathcal{J}$  will be contained in a forthcoming paper [67].

### 3. $H^\infty$ -CALCULI AND (STOCHASTIC) MAXIMAL $L^p$ -REGULARITY

Let  $A$  be a sectorial operator, or equivalently, let  $-A$  be the generator of a bounded analytic  $C_0$ -semigroup  $S = (S(t))_{t \geq 0}$  of bounded linear operators on a Banach space  $X$ . As is well known (see [2, Proposition I.1.4.1]), the spectrum of  $A$  is contained in the closure of a sector

$$\Sigma_\vartheta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \vartheta\}$$

for some  $\vartheta \in (0, \frac{1}{2}\pi)$ , and for all  $\sigma \in (\vartheta, \pi)$  one has

$$(3.1) \quad \sup_{z \in \mathbb{C} \setminus \Sigma_\sigma} \|z(z - A)^{-1}\| < \infty.$$

In the converse direction, this property characterize negative generators of bounded analytic  $C_0$ -semigroups. We refer to [30, 73] for more proofs and further results.

For  $\alpha \in (0, 1)$  we write

$$D_A(\alpha, p) = X_{\alpha, p} = (X, D(A))_{\alpha, p}, \quad X_\alpha = [X, D(A)]_\alpha$$

for the real and complex interpolation scales associated with  $A$ . If  $A$  has bounded imaginary powers, then (see [35, Theorem 6.6.9], [84, Theorem 1.15.3])

$$(3.2) \quad X_\alpha = \mathbf{D}(A^\alpha) \quad \text{with equivalent norms.}$$

The results of Section 4 and Subsection 5.1 are of isomorphic nature and the choice of the norm on  $X_\alpha$  is immaterial. In Subsection 5.3 we shall present a sharp result which is of isometric nature, for which it is important to work with the homogeneous norm on  $X_\alpha$  (assuming bounded invertibility of  $A$ ). We return to this point in Subsection 5.3.

We will need the following result (see [84, Theorem 1.14.5]).

**Proposition 3.1.** *– $A$  be the generator of a bounded analytic  $C_0$ -semigroup  $S = (S(t))_{t \geq 0}$  of bounded linear operators on a Banach space  $X$ , and suppose that  $0 \in \varrho(A)$ . For  $x \in X$  the following assertions are equivalent:*

- (1) *The orbit  $t \mapsto S(t)x$  belongs to  $H^{1,p}(\mathbb{R}_+; X) \cap L^p(\mathbb{R}_+; \mathbf{D}(A))$ .*
- (2) *The vector  $x$  belongs to  $\mathbf{D}_A(1 - \frac{1}{p}, p)$ .*

*If these equivalent conditions hold, then for all  $x \in \mathbf{D}_A(1 - \frac{1}{p}, p)$  one has*

$$\max \{ \|t \mapsto S(t)x\|_{H^{1,p}(\mathbb{R}_+; X)}, \|t \mapsto S(t)x\|_{L^p(\mathbb{R}_+; \mathbf{D}(A))} \} \approx \|x\|_{\mathbf{D}_A(1 - \frac{1}{p}, p)}.$$

**3.1. Operators with bounded  $H^\infty$ -calculus.** Let  $H^\infty(\Sigma_\sigma)$  denote the Banach space of all bounded analytic functions  $\varphi : \Sigma_\sigma \rightarrow \mathbb{C}$  endowed with the supremum norm. Let  $H_0^\infty(\Sigma_\sigma)$  be its linear subspace consisting of all functions satisfying an estimate

$$|\varphi(z)| \leq \frac{C|z|^\varepsilon}{(1 + |z|^2)^\varepsilon}$$

for some  $\varepsilon > 0$ .

Now let  $-A$  be as above and define, for  $\varphi \in H_0^\infty(\Sigma_\sigma)$  and  $\sigma < \sigma' < \pi$ ,

$$\varphi(A) = \frac{1}{2\pi i} \int_{\partial \Sigma_{\sigma'}} \varphi(z)(z - A)^{-1} dz.$$

This integral converges absolutely and is independent of  $\sigma'$ . We say that  $A$  has a *bounded  $H^\infty(\Sigma_\sigma)$ -calculus* if there is a constant  $C \geq 0$  such that

$$(3.3) \quad \|\varphi(A)\| \leq C\|\varphi\|_\infty \quad \forall \varphi \in H_0^\infty(\Sigma_\sigma).$$

The least constant  $C$  for which this holds will be referred to as the *boundedness constant* of the  $H^\infty(\Sigma_\sigma)$ -calculus. By approximation, the estimate (3.3) can be extended to all functions  $f \in H^\infty(\Sigma_\sigma)$ . The infimum of all  $\sigma$  such that  $A$  admits a bounded  $H^\infty(\Sigma_\sigma)$ -calculus is called the *angle* of the calculus.

Any operator  $A$  with a bounded  $H^\infty$ -calculus of angle less than  $\frac{1}{2}\pi$  had bounded imaginary powers. In particular, (3.2) applies to such operators.

We proceed with some examples of operators  $-A$  for which  $A$  has a bounded  $H^\infty$ -calculus of angle  $< \frac{1}{2}\pi$ ; we refer to [22, 54, 87] for further references.

*Example 3.2.*

- (1) Generators of analytic  $C_0$ -contraction semigroups on Hilbert spaces [59].
- (2) Generators of bounded analytic  $C_0$ -semigroups admitting Gaussian bounds [27].
- (3) Generators of positive analytic  $C_0$ -contraction semigroups on a space  $L^q(\mu)$ ,  $1 < q < \infty$  [41].

- (4) Second order uniformly elliptic operators [3, 21] on  $L^q(\mathbb{R}^d)$  and on  $L^q(\mathcal{O})$  for bounded  $C^2$ -domains  $\mathcal{O} \subseteq \mathbb{R}^d$  (with Dirichlet or Neumann boundary conditions) [3, 21].
- (5) The Stokes operator associated with the Navier-Stokes equation on bounded domains [40, 71] (see Section 8) and on unbounded domains [51].
- (6) Suppose  $-A$  generates a symmetric submarkovian  $C_0$ -semigroup  $S$  on a space  $L^2(\mu)$ . Then, for all  $q \in (1, \infty)$ ,  $A$  admits a bounded  $H^\infty$ -calculus of angle  $< \frac{1}{2}\pi$  on  $L^q(\mu)$  [53].
- (7) The Laplace-Beltrami operator  $-A := \Delta_{\text{LB}}$  on a complete Riemannian manifold  $M$  is given by the symmetric Dirichlet form  $-\langle \Delta_{\text{LB}} f, g \rangle = \int_M \nabla f \cdot \nabla g$  and therefore it satisfies the assumptions of example (6) [5, 82].
- (8) Let  $\gamma$  denote the standard Gaussian measure on  $\mathbb{R}^n$ . The Ornstein-Uhlenbeck operator  $-A = \Delta_{\text{OU}} := \Delta - x \cdot \nabla$  on satisfies the assumptions of example (6). This example admits various generalisations; see [16, 80] (for the infinite-dimensional symmetric case) [60] (for the finite-dimensional non-symmetric case) and [58] (for the infinite-dimensional non-symmetric case).

In example (4), under mild assumptions of the coefficients one typically has

$$\mathbf{D}(A^{\frac{1}{2}}) = H^{1,q}(\mathbb{R}^d) \quad \text{and} \quad H_{\text{Dir/Neum}}^{1,q}(\mathcal{O})$$

respectively (see, e.g., [35, Proposition 3.1.7] and the references in Sections 6 and 7). If, in example (7), the Ricci curvature of  $M$  is bounded below, then

$$\mathbf{D}((-\Delta_{\text{LB}})^{\frac{1}{2}}) = H^{1,q}(M),$$

the first order Sobolev space associated with the derivative  $\nabla$  [5]. In example (8), the classical Meyer inequalities imply that

$$\mathbf{D}((-\Delta_{\text{OU}})^{\frac{1}{2}}) = \mathbb{D}^{1,q}(\mathbb{R}^n, \gamma),$$

the first order Sobolev space associated with the Malliavin derivative in  $L^q(\mathbb{R}^n; \gamma)$  [72]. Necessary and sufficient conditions for the validity of the analogous identification in the non-symmetric and infinite-dimensional case were obtained in [58]; special cases were obtained earlier in [16, 60, 80].

**3.2. Maximal  $L^p$ -regularity.** Let  $-A$  be the generator of a bounded analytic  $C_0$ -semigroup  $S$  on a Banach space  $X$ . For functions  $g \in L^1_{\text{loc}}(\mathbb{R}_+; X)$  we consider the linear inhomogeneous problem

$$(3.4) \quad \begin{cases} u'(t) + Au(t) = g(t), & t > 0, \\ u(0) = 0. \end{cases}$$

The (unique) mild solution to (3.4) is given by

$$u(t) = S * g(t) := \int_0^t S(t-s)g(s) ds.$$

Let  $p \in (1, \infty)$ . For functions  $g \in L^p(\mathbb{R}_+; X)$ , a routine estimate shows that for all  $\delta \in [0, 1)$ ,  $S * g$  takes values in  $\mathbf{D}(A^\delta)$  almost everywhere on  $\mathbb{R}_+$ . The operator  $A$  has *maximal  $L^p$ -regularity* if for all  $g \in L^p(\mathbb{R}_+; X)$  the mild solution  $u$  belongs to  $\mathbf{D}(A)$  almost everywhere on  $\mathbb{R}_+$ , and satisfies

$$(3.5) \quad \|Au\|_{L^p(\mathbb{R}_+; X)} \leq C \|g\|_{L^p(\mathbb{R}_+; X)},$$

where  $C$  is a constant independent of  $g$ . If  $A$  has maximal  $L^p$ -regularity, then the mild solution  $u$  satisfies the identity

$$u(t) = u_0 + \int_0^t Au(s) ds + \int_0^t g(s) ds,$$

and the Lebesgue differentiation theorem shows that  $u$  is differentiable almost everywhere on  $\mathbb{R}_+$  with derivative  $u'(t) = Au(t) + g(t)$ . As a consequence, the inequality (3.5) self-improves to

$$(3.6) \quad \|u'\|_{L^p(\mathbb{R}_+; X)} + \|Au\|_{L^p(\mathbb{R}_+; X)} \leq C\|g\|_{L^p(\mathbb{R}_+; X)},$$

with a possibly different constant  $C$ .

In the definition of maximal  $L^p$ -regularity we do not insist that  $u$  itself be in  $L^p(\mathbb{R}_+; X)$ . If, however,  $0 \in \varrho(A)$ , then  $Au \in L^p(\mathbb{R}_+; X)$  implies  $u \in L^p(\mathbb{R}_+; X)$ , and the estimate (3.6) is then equivalent to

$$\|u\|_{H^{1,p}(\mathbb{R}_+; X)} + \|u\|_{L^p(\mathbb{R}_+; D(A))} \leq C\|g\|_{L^p(\mathbb{R}_+; X)}.$$

The following result was proved in [86] (part (1)) and [42] (part (2)); the final assertion follows by standard trace and interpolation techniques (see [2, Theorem III.4.10.2]).

**Theorem 3.3.** *Let  $-A$  be the generator of an analytic  $C_0$ -semigroup on a UMD space  $X$ .*

- (1) *The operator  $A$  has a maximal  $L^p$ -regularity for some (equivalently, all)  $p \in (1, \infty)$  if and only if the set  $\{\lambda(\lambda + A)^{-1} : \lambda \in i\mathbb{R} \setminus \{0\}\}$  is  $R$ -bounded in  $\mathcal{L}(X)$ .*
- (2) *If  $A$  has a bounded  $H^\infty$ -calculus of angle  $< \frac{1}{2}\pi$ , then  $A$  has maximal  $L^p$ -regularity for all  $p \in (1, \infty)$ .*

*If  $A$  has maximal  $L^p$ -regularity and  $0 \in \varrho(A)$ , then the mild solution  $u = S * g$  of (3.4) belongs to  $BUC(\mathbb{R}_+; D_A(1 - \frac{1}{p}, p))$  and*

$$\|u\|_{BUC(\mathbb{R}_+; D_A(1 - \frac{1}{p}, p))} \leq C\|g\|_{L^p(\mathbb{R}_+; X)}$$

*with a constant  $C$  independent of  $g$ .*

**3.3. Stochastic maximal  $L^p$ -regularity.** In this section we assume that  $-A$  generates a bounded analytic  $C_0$ -semigroup on a UMD space  $X$  with type 2. For processes  $G \in L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$  we consider the problem

$$\begin{cases} dU(t) + AU(t) dt = G(t) dW_H(t), & t > 0, \\ U(0) = 0. \end{cases}$$

The (unique) mild solution of this problem is given by

$$U(t) = \int_0^t S(t-s) G(s) dW_H(s).$$

Note that this stochastic integral is well defined in view of (2.5) and the remark following it. A routine estimate based on (2.5) and Young's inequality shows that for all  $\delta \in [0, \frac{1}{2})$ ,  $U$  takes values in  $D(A^\delta)$  almost everywhere on  $\mathbb{R}_+ \times \Omega$ . The operator  $A$  is said to have *stochastic maximal  $L^p$ -regularity* if for all  $G \in L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ ,  $U$  belongs to  $D(A^{\frac{1}{2}})$  almost everywhere on  $\mathbb{R}_+ \times \Omega$  and satisfies

$$(3.7) \quad \|A^{\frac{1}{2}}U\|_{L^p(\mathbb{R}_+ \times \Omega; X)} \leq C\|G\|_{L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))}.$$

with a constant  $C$  independent of  $G$ . Under the additional assumption  $0 \in \varrho(A)$ ,  $A^{\frac{1}{2}}U \in L^p(\mathbb{R}_+ \times \Omega; X)$  implies  $U \in L^p(\mathbb{R}_+ \times \Omega; X)$  and (3.7) is equivalent to

$$(3.8) \quad \|U\|_{L^p(\mathbb{R}_+ \times \Omega; \mathcal{D}(A^{\frac{1}{2}}))} \leq C \|G\|_{L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))}.$$

*Remark 3.4.* It follows from [68] that  $A$  has stochastic maximal  $L^p$ -maximal regularity if and only if (3.7) holds for all deterministic  $G \in L^p(\mathbb{R}_+; \gamma(H, X))$ . For later use we note that by Theorem 2.3, this condition is equivalent to

$$(3.9) \quad \int_0^\infty \|s \mapsto A^{\frac{1}{2}}S(t-s)G(s)\|_{\gamma(L^2(0,t;H),X)}^p dt \leq C^p \|G\|_{L^p(\mathbb{R}_+; \gamma(H, X))}^p.$$

Comparing the notions of deterministic maximal  $L^p$ -regularity and stochastic maximal  $L^p$ -regularity, we note that the latter increases the regularity only by an exponent  $\frac{1}{2}$ . Another difference is that stochastic maximal  $L^p$ -regularity does not in general imply  $u \in H^{\frac{1}{2},p}(\mathbb{R}_+; L^p(\Omega; X))$  (see, however, (3.11) for a related result which does hold true). In fact (this corresponds to the case  $H = X = \mathbb{R}$ ,  $A = 0$ , and  $G$  constant), already Brownian motions fail to belong to  $H^{\frac{1}{2},p}(0, 1; L^p(\Omega))$  for any  $p \in [1, \infty]$ . This follows from the continuous inclusion

$$H^{\frac{1}{2},p}(0, 1; L^p(\Omega)) \simeq L^p(\Omega; H^{\frac{1}{2},p}(0, 1)) \hookrightarrow L^p(\Omega; B_{p,p \wedge 2}^{\frac{1}{2}}(0, 1))$$

and the results in [17, 37].

Recall the operator family  $\mathcal{J}$  which has been introduced in (2.6). By Theorem 2.5, the  $R$ -boundedness of  $\mathcal{J}$  is satisfied if  $X$  is isomorphic to a closed subspace of an  $L^q$ -space.

The next theorem has been proved in [68, Theorems 1.1, 1.2] for spaces  $X = L^q(\mu)$  with  $q \geq 2$  and  $\mu$   $\sigma$ -finite. Inspection of the proof shows that it consists of two parts: (i) the proof that  $\mathcal{J}$  is  $R$ -bounded for such  $X = L^q(\mu)$  ([68, Theorem 3.1], recalled here as Theorem 2.5) and (ii) the proof that, still for  $X = L^q(\mu)$ , the  $R$ -boundedness of  $\mathcal{J}$  implies the result. Step (ii) extends *mutatis mutandis* to arbitrary UMD Banach space with type 2, provided one replaces spaces of square functions such as  $L^q(\mu; \mathcal{H})$  and duality for Hilbert spaces  $\mathcal{H}$  by spaces of radonifying operators  $\gamma(H, X)$  and trace duality following the lines of [42]. This leads to the following result:

**Theorem 3.5** (Conditions for stochastic maximal  $L^p$ -regularity). *Let  $X$  be a UMD space with type 2 and let  $p \in [2, \infty)$ , and suppose the operator family  $\mathcal{J}$  is  $R$ -bounded from  $\mathcal{L}(L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X)))$  to  $L^p(\mathbb{R}_+ \times \Omega; X)$ . If  $A$  has a bounded  $H^\infty$ -calculus on  $X$  of angle  $< \frac{1}{2}\pi$ , then  $A$  has stochastic maximal  $L^p$ -regularity. If, in addition,  $0 \in \varrho(A)$ , then also (3.8) holds and*

$$(3.10) \quad \|U\|_{L^p(\Omega; BUC(\mathbb{R}_+; \mathcal{D}_A(\frac{1}{2} - \frac{1}{p}, p))} \leq C \|G\|_{L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))}.$$

and, for all  $\theta \in [0, \frac{1}{2})$ ,

$$(3.11) \quad \|U\|_{L^p(\Omega; H^{\theta,p}(\mathbb{R}_+; \mathcal{D}(A^{\frac{1}{2}-\theta})))} \leq C \|G\|_{L^p(\mathbb{R}_+ \times \Omega; \gamma(H, X))}.$$

In all these estimates, the constants  $C$  are independent of  $G$ .

Note that the case  $\theta = 0$  of (3.11) corresponds to the stochastic maximal  $L^p$ -regularity estimate (3.7). The proof of (3.11) proceeds by reducing the problem, via the  $H^\infty$ -calculus of  $A$ , to the  $R$ -boundedness of a certain family  $\mathcal{J}$  of stochastic convolution operators with scalar-valued kernels. By convexity arguments, the  $R$ -boundedness of  $\mathcal{J}$  is then deduced from the  $R$ -boundedness of  $\mathcal{J}$ . The estimate



- (b) there exist constants  $L_B, \tilde{L}_B, C_B$  such that for all  $t \in [0, T], \omega \in \Omega$ , and  $x, y \in X_1$ ,

$$\|B(t, \omega, x) - B(t, \omega, y)\|_{\gamma(H, X_{\frac{1}{2}})} \leq L_B \|x - y\|_{X_1} + \tilde{L}_B \|x - y\|_{X_0}$$

and

$$\|B(t, \omega, x)\|_{\gamma(H, X_{\frac{1}{2}})} \leq C_B(1 + \|x\|_{X_1}).$$

(Hu<sub>0</sub>) The initial value  $u_0 : \Omega \rightarrow X_0$  is strongly  $\mathcal{F}_0$ -measurable.

*Remark 4.1.* Some comments on these assumptions are in order.

- (i) By (HA), the spaces  $X_0$  and  $X_1$  are isomorphic as Banach spaces, an isomorphism being given by  $(\lambda - A(\omega))^{-1}$  for any  $\lambda \in \varrho(A(\omega))$ . In particular, since  $X_0$  is a UMD space with type 2, the same is true for  $X_1$ . As a consequence, also the real and complex interpolation spaces  $X_{\alpha, p}$  with  $p \in [2, \infty)$  and  $X_\alpha$  are UMD spaces with type 2 (see [39, Proposition 5.1]).
- (ii) If (HA) holds for some  $w \in \mathbb{R}$ , then it holds for any  $w' > w$ . Furthermore, we may write

$$-A + F = -(A + w') + (F + w'),$$

and note that a function  $F$  satisfies (HF) if and only if  $F + w'$  satisfies (HF). Thus, in what follows we may replace  $A$  and  $F$  by  $A + w'$  and  $F + w'$  and thereby assume, without any loss of generality, that the operators  $A(\omega)$  are *invertible*, uniformly in  $\omega$ .

- (iii) The operators  $-A(\omega)$  generate analytic  $C_0$ -semigroups  $S(\omega)$  on  $X_0$ , given through the  $H^\infty$ -calculus by

$$S(t, \omega) = e^{-tA(\omega)}, \quad t \geq 0.$$

For each  $t \geq 0$  and  $x \in X_0$ ,  $\omega \mapsto S(t, \omega)x$  is strongly  $\mathcal{F}_0$ -measurable. Assuming, as in (ii), that the operators  $A(\omega)$  are uniformly invertible, the semigroups  $S(\cdot, \omega)$  are uniformly exponentially stable, uniformly in  $\omega$ .

- (iv) By (3.9), Theorem 3.5 extends to the present situation of a random operator  $A$  satisfying (HA).
- (v) The Lipschitz conditions in (HF) and (HB) are fulfilled if and only if there exist  $\alpha_F, \alpha_B \in [0, 1)$  and constants  $L'_F, \tilde{L}'_F, L'_B, \tilde{L}'_B$  such that

$$\|F(t, \omega, x) - F(t, \omega, y)\|_{X_0} \leq L'_F \|x - y\|_{X_1} + \tilde{L}'_F \|x - y\|_{X_{\alpha_F}}$$

and

$$\|B(t, \omega, x) - B(t, \omega, y)\|_{\gamma(H, X_{\frac{1}{2}})} \leq L'_B \|x - y\|_{X_1} + \tilde{L}'_B \|x - y\|_{X_{\alpha_B}}.$$

Moreover, for any  $\varepsilon > 0$  the constants  $\tilde{L}'_F$  and  $\tilde{L}'_B$  can be chosen in such a way that  $|L'_F - L_F| < \varepsilon$  and  $|L'_B - L_B| < \varepsilon$ . The ‘if’ part is obvious from  $\|x - y\|_{X_0} \lesssim_{\alpha} \|x - y\|_{X_\alpha}$  (in this case we may take  $L'_F = L_F$  and  $L'_B = L_B$ ), and the ‘only if’ part follows by a standard application of Young’s inequality. Indeed, for any  $\delta > 0$  we have

$$\|x - y\|_{X_\alpha} \leq C \|x - y\|_{X_0}^{1-\alpha} \|x - y\|_{X_1}^\alpha \leq \frac{C}{(1-\alpha)\delta} \|x - y\|_{X_0} + \frac{C\delta}{\alpha} \|x - y\|_{X_1}.$$

Choosing  $\delta > 0$  small enough this gives the required result. In certain applications (see Sections 6, 7 and 8 below) this reformulation of the conditions (HF) and (HB) is more convenient.

**Definition 4.2.** *Let (H) be satisfied. A process  $U : [0, T] \times \Omega \rightarrow X_0$  is called a strong solution of (SE) if it is strongly measurable and adapted, and*

- (i) *almost surely,  $U \in L^2(0, T; X_1)$ ;*
- (ii) *for all  $t \in [0, T]$ , almost surely the following identity holds in  $X_0$ :*

$$U(t) + \int_0^t AU(s) ds = u_0 + \int_0^t F(s, U(s)) + f(s) ds + \int_0^t B(s, U(s)) + b(s) dW_H(s).$$

To see that the integrals in this definition are well defined, we note that, by (HA), the process  $AU$  is strongly measurable and satisfies

$$\|AU\|_{L^1(0, T; X_0)} \leq \|A\|_{\mathcal{L}(X_1, X_0)} \|U\|_{L^1(0, T; X_1)}$$

almost surely. Similarly, by (HF) and (HB),  $F(\cdot, U)$  and  $f$  belong to  $L^1(0, T; X_0)$  and  $B(\cdot, U)$  and  $b$  belong to  $L^2(0, T; \gamma(H, X_{\frac{1}{2}}))$  almost surely. Therefore, the Bochner integral is well defined in  $X_0$ , and the stochastic integral is well defined in  $X_{\frac{1}{2}}$  (and hence in  $X_0$ ) by (HX), the fact the space  $X_{\frac{1}{2}}$  is a UMD space with type 2, and (2.5).

By Definition 4.2, a strong solution always has a version with continuous paths in  $X_0$  such that, almost surely, the identity in (ii) holds for all  $t \in [0, T]$ . Indeed, define  $\tilde{U} : [0, T] \times \Omega \rightarrow X_0$  by

$$\tilde{U}(t) := - \int_0^t AU(s) ds + u_0 + \int_0^t F(s, U(s)) + f(s) ds + \int_0^t B(s, U(s)) + b(s) dW_H(s),$$

where we take continuous versions of the integrals on the right-hand side. From the definitions of  $U$  and  $\tilde{U}$  one obtains, for all  $t \in [0, T]$ , that  $U(t) = \tilde{U}(t)$  almost surely in  $X_0$ . Therefore, almost surely, for all  $t \in [0, T]$  one has

$$\tilde{U}(t) + \int_0^t A\tilde{U}(s) ds = u_0 + \int_0^t F(s, \tilde{U}(s)) + f(s) ds + \int_0^t B(s, \tilde{U}(s)) + b(s) dW_H(s).$$

From now on we choose this version whenever this is convenient. We will actually prove much stronger regularity properties in Theorem 4.5 below.

**Definition 4.3.** *Let (H) be satisfied. A process  $U : [0, T] \times \Omega \rightarrow X_0$  is called a mild solution of (SE) if it is strongly measurable and adapted, and*

- (i) *almost surely,  $U \in L^2(0, T; X_1)$ ;*
- (ii) *for all  $t \in [0, T]$ , almost surely the following identity holds in  $X_0$ :*

$$U(t) = S(t)u_0 + \int_0^t S(t-s)[F(s, U(s)) + f(s)] ds + \int_0^t S(t-s)[B(s, U(s)) + b(s)] dW_H(s).$$

The convolutions with  $F(\cdot, U(\cdot))$  and  $f$  are well defined as an  $X_0$ -valued process by (HF). The stochastic convolutions with  $B(\cdot, U(\cdot))$  and  $b$  are well defined as an  $X_{\frac{1}{2}}$ -valued process (and hence as an  $X_0$ -valued process) by (HB), the fact that  $X_{\frac{1}{2}}$  is a UMD space with type 2, and (2.5). Henceforth we shall use the notations

$$S * g(t) := \int_0^t S(t-s)g(s) ds,$$

$$S \diamond G(t) := \int_0^t S(t-s)G(s) dW_H(s),$$

whenever the integrals are well defined.

**Proposition 4.4.** *Let (H) be satisfied. A process  $U : [0, T] \times \Omega \rightarrow X_0$  is a strong solution of (SE) if and only if it is a mild solution of (SE).*

Results of this type for time-dependent operators  $A$  are well known. Since in our case  $A$  also depends on  $\Omega$ , the usual proof has to be adjusted. For the reader's convenience we provide the details.

*Proof.* For notational convenience we write  $\bar{F}(t, x) = F(t, x) + f(t)$  and  $\bar{B}(t, x) = B(t, x) + b(t)$ .

First assume that  $U$  is a mild solution. As in [19, Proposition 6.4 (i)], the (stochastic) Fubini theorem can be used to show that for all  $t \in [0, T]$ , almost surely we have

$$U(t) + \int_0^t AU(s) ds = u_0 + \int_0^t \bar{F}(s, U(s)) ds + \int_0^t \bar{B}(s, U(s)) dW_H(s).$$

Next assume that  $U$  is a strong solution of (SE). By the scalar-valued Itô formula,

$$\begin{aligned} \langle U(t), \varphi(t) \rangle - \langle u_0, \varphi(0) \rangle &= \int_0^t \langle AU(s), \varphi(s) \rangle + \langle U(s), \varphi'(s) \rangle ds \\ &\quad + \int_0^t \langle \bar{F}(s, U(s)), \varphi(s) \rangle ds \\ &\quad + \int_0^t \bar{B}(s, U(s))^* \varphi(s) dW_H(s), \end{aligned}$$

for functions  $\varphi \in C^1([0, t]; E^*)$  of the form  $\varphi = g \otimes x^*$ . By linearity and density this extends to all  $\varphi \in C^1([0, t]; E^*)$ . By linearity and approximation this extends to all  $\varphi \in L^0(\Omega; C^1([0, t]; E^*))$  which are  $\mathcal{F}_0$ -measurable. Indeed, recall that for a  $\lim_{n \rightarrow \infty} \int_0^t \psi(t) - \psi_n(t) dW(t) = 0$  in  $L^0(\Omega; C([0, T]))$  whenever  $\lim_{n \rightarrow \infty} \psi_n = \psi$  in  $L^0(\Omega; L^2(0, T; H))$  (see [38, Proposition 17.6]).

With the choice  $\varphi(t) = S^*(t-s)\lambda(\lambda + A^*)^{-1}x^*$  we obtain, for all  $x^* \in E^*$  and  $\lambda > w$  (with  $w$  as in (HA)),

$$\begin{aligned} &\langle \lambda(\lambda + A)^{-1}U(t), x^* \rangle - \langle \lambda(\lambda + A)^{-1}S(t)u_0, x^* \rangle \\ &= \left\langle \lambda(\lambda + A)^{-1} \int_0^t \langle S(t-s)\bar{F}(s, U(s)) ds, x^* \rangle \right. \\ &\quad \left. + \left\langle \lambda(\lambda + A)^{-1} \int_0^t S(t-s)\bar{B}(s, U(s)) dW_H(s), x^* \right\rangle, \right. \end{aligned}$$

where we used the strong  $\mathcal{F}_0$ -measurability of  $A$ . Now the result follows from the fact that for all  $x \in X$ ,  $\lambda(\lambda + A)^{-1}x \rightarrow x$  as  $\lambda \rightarrow \infty$ .  $\square$

Let us fix an exponent  $p \in [2, \infty)$  for the moment and assume, as in Remark 4.1(ii), that the operators  $A(\omega)$  are uniformly invertible. By Theorem 3.3 and (HX) and (HA), the linear operator

$$g \mapsto S * g$$

is bounded from  $L^p_{\mathcal{F}}(\Omega; L^p(\mathbb{R}_+; X_0))$  into  $L^p_{\mathcal{F}}(\Omega; L^p(\mathbb{R}_+; X_1))$ . Furthermore, if the operator family  $\mathcal{J}$  introduced in Subsection 2.1.4 is  $R$ -bounded in

$$\mathcal{L}(L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X_0)), L^p(\mathbb{R}_+ \times \Omega; X_0)),$$

then it is also  $R$ -bounded in

$$\mathcal{L}(L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X_{\frac{1}{2}})), L^p(\mathbb{R}_+ \times \Omega; X_{\frac{1}{2}}))$$

and therefore by Theorem 3.5 (applied to the space  $X_{\frac{1}{2}}$ ) and (HX) and (HA), the reiteration identity  $X_1 = (X_{\frac{1}{2}})_{\frac{1}{2}}$  (apply  $A^{\frac{1}{2}}$  to both sides and use that  $D(A^{\frac{1}{2}}) = X_{\frac{1}{2}}$  by (HA)) the mapping

$$G \mapsto S \diamond G$$

is bounded from  $L^p_{\mathcal{F}}(\Omega; L^p(\mathbb{R}_+; \gamma(H, X_{\frac{1}{2}})))$  into  $L^p_{\mathcal{F}}(\Omega; L^p(\mathbb{R}_+; X_1))$ . We shall denote by

$$K_p^* \text{ and } K_p^\diamond$$

the norms of these operators. We emphasise that the numerical value of these constants depends on the choice of the parameter  $w'$  used for rescaling  $A$  (cf. remark 4.1).

In what follows we fix an arbitrary time horizon  $T > 0$ ; constants appearing in the inequalities below are allowed to depend on it. Recall that by Theorem 2.5, the  $R$ -boundedness of the operator family  $\mathcal{J}$  is satisfied if  $X_0$  is isomorphic to a closed subspace of an  $L^q$ -space.

**Theorem 4.5.** *Let (H) be satisfied, let  $p \in [2, \infty)$ , let  $f \in L^p_{\mathcal{F}}(\Omega; L^p(0, T; X_0))$  and  $b \in L^p_{\mathcal{F}}(\Omega; L^p(0, T; \gamma(H, X_{\frac{1}{2}})))$ , and suppose that the operator family  $\mathcal{J}$  is  $R$ -bounded from  $\mathcal{L}(L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X_0))$  to  $L^p(\mathbb{R}_+ \times \Omega; X_0)$ . If the Lipschitz constants  $L_F$  and  $L_B$  satisfy*

$$K_p^* L_F + K_p^\diamond L_B < 1,$$

then the following assertions hold:

- (i) *If  $u_0 \in L^0_{\mathcal{F}_0}(\Omega; X_{1-\frac{1}{p}, p})$ , then the problem (SE) has a unique strong solution  $U$  in*

$$L^0_{\mathcal{F}}(\Omega; L^p(0, T; X_1)) \cap L^0_{\mathcal{F}}(\Omega; C([0, T]; X_{1-\frac{1}{p}, p})).$$

- (ii) *If  $u_0 \in L^p_{\mathcal{F}_0}(\Omega; X_{1-\frac{1}{p}, p})$ , then the strong solution  $U$  given by part (i) belongs to*

$$L^p_{\mathcal{F}}((0, T) \times \Omega; X_1) \cap L^p_{\mathcal{F}}(\Omega; C([0, T]; X_{1-\frac{1}{p}, p}))$$

and satisfies

$$\|U\|_{L^p((0, T) \times \Omega; X_1)} \leq C(1 + \|u_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}),$$

$$\|U\|_{L^p(\Omega; C([0, T]; X_{1-\frac{1}{p}, p}))} \leq C(1 + \|u_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}),$$

with constants  $C$  independent of  $u_0$ .

(iii) For all  $u_0, v_0 \in L^p_{\mathcal{F}_0}(\Omega; X_{1-\frac{1}{p}, p})$ , the corresponding strong solutions  $U, V$  satisfy

$$\begin{aligned} \|U - V\|_{L^p((0, T) \times \Omega; X_1)} &\leq C \|u_0 - v_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}, \\ \|U - V\|_{L^p(\Omega; C([0, T]; X_{1-\frac{1}{p}, p}))} &\leq C \|u_0 - v_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}, \end{aligned}$$

with constants  $C$  independent of  $u_0$  and  $v_0$ .

*Remark 4.6.* The condition  $u_0 \in L^0_{\mathcal{F}_0}(\Omega; X_{1-\frac{1}{p}, p})$  is satisfied if  $(Hu_0)$  holds and  $u_0$  takes values in  $X_{1-\frac{1}{p}, p}$  almost surely. Indeed, by  $(Hu_0)$  we know that  $u_0$  is strongly  $\mathcal{F}_0$ -measurable as an  $X$ -valued random variable. Now the strong  $\mathcal{F}_0$ -measurability of  $u_0$  as an  $X_{1-\frac{1}{p}, p}$ -valued random variable easily follows from the strong measurability of  $\xi : \Omega \rightarrow L^p(0, 1, \frac{dt}{t}; X)$ , given by

$$\xi(\omega) := [t \mapsto AS(t)u_0(\omega)],$$

and the definition of  $X_{1-\frac{1}{p}, p}$ .

*Proof of Theorem 4.5.* Without loss of generality we can reduce to the case where  $w = 0$  (see Remark 4.1 (ii)). By assumption we have  $K_p^* L_F + K_p^\diamond L_B = 1 - \theta$  for some  $\theta \in (0, 1]$ . Without loss of generality we may assume that  $L_F + L_B > 0$  and  $\theta \in (0, 1)$ .

By Proposition 4.4 it suffices to prove existence and uniqueness of a mild solution.

*Step 1:* Local existence of mild solutions for initial values  $u_0 \in L^p_{\mathcal{F}_0}(\Omega; X_{1-\frac{1}{p}, p})$ . We fix a number  $\kappa \in (0, T]$ , to be chosen in a moment, and introduce, for  $\theta \in [0, 1]$ , the Banach spaces

$$\begin{aligned} Z_{\theta, \kappa} &= L^p_{\mathcal{F}}(\Omega; L^p(0, \kappa; X_\theta)), \\ Z_{\theta, \kappa}^\gamma &= L^p_{\mathcal{F}}(\Omega; L^p(0, \kappa; \gamma(H, X_\theta))). \end{aligned}$$

On  $Z_{1, \kappa}$  we define an equivalent norm  $\|\cdot\|$  by

$$\|\phi\| = \|\phi\|_{Z_{1, \kappa}} + M \|\phi\|_{Z_{0, \kappa}}$$

with  $M = (K_p^* \tilde{L}_F + K_p^\diamond \tilde{L}_B) / (K_p^* L_F + K_p^\diamond L_B)$ .

In order to simplify notations we shall omit the subscript  $\kappa$  in what follows. Let  $L : Z_1 \rightarrow Z_1$  be the mapping given by

$$L(\phi)(t) = S(t)u_0 + S * [F(\cdot, \phi) + f](t) + S \diamond [B(\cdot, \phi) + b](t).$$

We emphasise that  $L$  depends on the initial value  $u_0$ .

First we check that  $L$  does indeed map  $Z_1$  into itself. By  $(Hu_0)$  and Proposition 3.1,  $t \mapsto S(t)u_0$  defines an element of  $Z_1$ .

By restriction to the interval  $[0, \kappa]$ , the operators  $g \mapsto S * g$  and  $G \mapsto S \diamond G$  are bounded as mappings from  $L^p_{\mathcal{F}}(\Omega; L^p(0, \kappa; X_0))$  and  $L^p_{\mathcal{F}}(\Omega; L^p(0, \kappa; \gamma(H, X_{\frac{1}{2}})))$  into  $L^p_{\mathcal{F}}(\Omega; L^p(0, \kappa; X_1))$ , with norms bounded by  $K_p^*$  and  $K_p^\diamond$  respectively. Therefore  $L$  is well defined as a mapping from  $Z_1$  into itself, and for all  $\phi_1, \phi_2 \in Z_1$  we may estimate

$$\begin{aligned} \|L(\phi_1) - L(\phi_2)\|_{Z_1} &\leq \|S * (F(\cdot, \phi_1) - F(\cdot, \phi_2))\|_{Z_1} + \|S \diamond (B(\cdot, \phi_1) - B(\cdot, \phi_2))\|_{Z_1} \\ &\leq K_p^* \|F(\cdot, \phi_1) - F(\cdot, \phi_2)\|_{Z_0} + K_p^\diamond \|B(\cdot, \phi_1) - B(\cdot, \phi_2)\|_{Z_{\frac{1}{2}}^\gamma} \\ &\leq K_p^* L_F \|\phi_1 - \phi_2\|_{Z_1} + K_p^* \tilde{L}_F \|\phi_1 - \phi_2\|_{Z_0} \end{aligned}$$

$$\begin{aligned}
& + K_p^\diamond L_B \|\phi_1 - \phi_2\|_{Z_1} + K_p^\diamond \tilde{L}_B \|\phi_1 - \phi_2\|_{Z_0} \\
& = (1 - \theta) \|\phi_1 - \phi_2\|,
\end{aligned}$$

recalling that  $K_p^* L_F + K_p^\diamond L_B = 1 - \theta$ . Moreover, we have the elementary estimate

$$\begin{aligned}
\|L(\phi_1) - L(\phi_2)\|_{Z_0} & \leq c(\kappa) [CL_F \|\phi_1 - \phi_2\|_{Z_1} + C\tilde{L}_F \|\phi_1 - \phi_2\|_{Z_0} \\
& \quad + C' L_B \|\phi_1 - \phi_2\|_{Z_1} + C' \tilde{L}_B \|\phi_1 - \phi_2\|_{Z_0}] \\
& \leq \tilde{c}(\kappa) \|\phi_1 - \phi_2\|,
\end{aligned}$$

where  $\kappa \mapsto c(\kappa)$  and  $\kappa \mapsto \tilde{c}(\kappa)$  are continuous functions on  $[0, T]$  not depending on  $u_0$  and satisfying  $\lim_{\kappa \downarrow 0} c(\kappa) = \lim_{\kappa \downarrow 0} \tilde{c}(\kappa) = 0$ .

Collecting the above estimates, we see that

$$\|L(\phi_1) - L(\phi_2)\| \leq (1 - \theta + M\tilde{c}(\kappa)) \|\phi_1 - \phi_2\|.$$

So far, the number  $\kappa > 0$  was arbitrary. Now we set

$$\kappa := \inf\{t \in (0, T] : M\tilde{c}(t) \geq \frac{1}{2}\theta\}.$$

where we take  $\kappa = T$  if the infimum is taken over the empty set. Note that  $\kappa$  only depends on  $\theta$ , the Lipschitz constants of  $F$  and  $B$ , the constants  $K_p^*$  and  $K_p^\diamond$  and the type 2 constant of  $X_{\frac{1}{2}}$ . Then  $(1 - \theta + M\tilde{c}(\kappa)) \leq 1 - \frac{1}{2}\theta$ , and it follows that  $L$  has a unique fixed point in  $Z_1$ . This gives a process  $U \in Z_1$  such that for almost all  $(t, \omega) \in [0, \kappa] \times \Omega$ , the following identity holds in  $X_1$ :

$$(4.1) \quad U(t) = S(t)u_0 + S * F(\cdot, U)(t) + S * f(t) + S \diamond B(\cdot, U)(t) + S \diamond b(t).$$

By Theorems 3.3 and 3.5 (applied with  $X = X_{\frac{1}{2}}$ ), and keeping in mind Remarks 4.1(i) and (iv),  $U$  has a version with trajectories in  $L^p(\Omega; C([0, \kappa], X_{1-\frac{1}{p}, p}))$ . For this version, almost surely the identity (4.1) holds in  $X_0$  for all  $t \in [0, \kappa]$ .

*Step 2:* Local existence of mild solutions for initial values  $u_0 \in L^0_{\mathcal{F}_0}(\Omega; X_{1-\frac{1}{p}, p})$ .

For  $n \geq 1$ , let

$$\Gamma_n := \{\|u_0\|_{X_{1-\frac{1}{p}, p}} \leq n\}.$$

From Step 1 we obtain processes  $U_n$  belonging to  $Z_1 \cap L^p(\Omega; C([0, \kappa], X_{1-\frac{1}{p}, p}))$  such that (4.1) holds with the pair  $(u_0, U)$  replaced by  $(u_{0,n}, U_n)$  (with  $u_{0,n} = \mathbf{1}_{\Gamma_n} u_0$ ). We claim that for all  $m \leq n$ ,  $U_n(\cdot, \omega) = U_m(\cdot, \omega)$  in  $X_{1-\frac{1}{p}, p}$  almost surely on  $\Gamma_m \times [0, \tau_m]$ . Indeed, by Step 1 and the fact that  $\Gamma_m \in \mathcal{F}_0$ ,

$$\begin{aligned}
\|\mathbf{1}_{\Gamma_m}(U_m - U_n)\| & = \|\mathbf{1}_{\Gamma_m}(L(U_m) - L(U_n))\| \\
& = \|\mathbf{1}_{\Gamma_m}(L(\mathbf{1}_{\Gamma_m} U_m) - L(\mathbf{1}_{\Gamma_m} U_n))\| \\
& \leq \|L(\mathbf{1}_{\Gamma_m} U_m) - L(\mathbf{1}_{\Gamma_m} U_n)\| \\
& \leq (1 - \frac{1}{2}\theta) \|\mathbf{1}_{\Gamma_m}(U_m - U_n)\|
\end{aligned}$$

and since  $\theta \in (0, 1)$  it follows that for almost all  $(t, \omega) \in [0, \kappa] \times \Gamma_n$ ,  $U_m(t, \omega) = U_n(t, \omega)$  in  $X_1$ . By (4.1) for  $U_m$  and  $U_n$  it follows that for almost all  $\omega \in \Gamma_m$ ,  $U_n(\cdot, \omega) = U_m(\cdot, \omega)$  in  $X_{1-\frac{1}{p}, p}$ , and the claim follows. Therefore, we can define  $U : [0, \kappa] \times \Omega \rightarrow X_0$  by  $U = U_n$  on  $\Gamma_n$ . Now it is easy to check that

$$U \in L^0_{\mathcal{F}}(\Omega; L^p(0, \kappa; X_1)) \cap L^0(\Omega; C([0, \kappa], X_{1-\frac{1}{p}, p})).$$

and that for all  $t \in [0, \kappa]$ , (4.1) holds almost surely in  $X_0$ .

*Step 3:* Local uniqueness of mild solutions for initial values  $u_0 \in L^0_{\mathcal{F}_0}(\Omega; X_{1-\frac{1}{p}, p})$ .

Let  $U, V \in L^0_{\mathcal{F}}(\Omega; L^p(0, \kappa; X_1))$  be such that (4.1) holds. For  $W \in \{U, V\}$  let  $\tau_n^W$  be the stopping time defined by

$$\tau_n^W = \inf\{t \in [0, \kappa] : \|\mathbf{1}_{[0,t]}W\|_{L^p(0,\kappa;X_1)} \geq n\}$$

(and  $\tau_n^W = \kappa$  if this set is empty) and let  $\tau_n = \tau_n^U \wedge \tau_n^V$ . Let  $U_n = \mathbf{1}_{[0,\tau_n]}U$  and  $V_n = \mathbf{1}_{[0,\tau_n]}V$ . Clearly, for all  $n \geq 1$ , we have  $U_n, V_n \in Z_1$ . Using the extension of [9, Lemma A.1] to the type 2 setting one can check that for all  $t \in [0, \kappa]$ , almost surely, one has

$$\begin{aligned} W_n &= \mathbf{1}_{[0,\tau_n]}S(\cdot)u_0 + \mathbf{1}_{[0,\tau_n]}(S * (\mathbf{1}_{[0,\tau_n]}(F(\cdot, W_n) + f))) \\ &\quad + \mathbf{1}_{[0,\tau_n]}(S \diamond (\mathbf{1}_{[0,\tau_n]}(B(\cdot, W_n) + b))) \end{aligned}$$

in  $X_0$ , where  $W_n \in \{U_n, V_n\}$ . As in Step 1 it follows that

$$\begin{aligned} &\|U_n - V_n\| \\ &\leq \|S * (\mathbf{1}_{[0,\tau_n]}(F(\cdot, U_n) - F(\cdot, V_n)))\| + \|S \diamond (\mathbf{1}_{[0,\tau_n]}(B(\cdot, U_n) - B(\cdot, V_n)))\| \\ &\leq (1 - \frac{1}{2}\theta)\|U_n - V_n\|. \end{aligned}$$

Since  $\theta \in (0, 1)$ , we obtain that  $U_n = V_n$  in  $Z_1$ . Letting  $n$  tend to infinity, we may conclude that  $U = V$  in  $L^0_{\mathcal{F}}(\Omega; L^p(0, \kappa; X_1))$ .

*Step 4: Global existence of mild solutions.*

In Steps 1 and 2 we have shown that there exists a unique mild solution  $U_1$  in  $L^0_{\mathcal{F}}(\Omega; L^p(0, \kappa; X_1))$  with trajectories in  $C([0, \kappa], X_{1-\frac{1}{p}, p})$ . Let, for  $0 \leq a < b \leq T$ ,

$$Y(a, b) := L^0_{\mathcal{F}}(\Omega; L^p(a, b; X_1)) \cap L^0(\Omega; C([a, b], X_{1-\frac{1}{p}, p}))$$

We construct a mild solution on  $[\kappa, 2\kappa]$ . Using the path continuity in  $X_{1-\frac{1}{p}, p}$  we can take  $u_\kappa = U_1(\kappa)$  in  $L^0(\Omega; X_{1-\frac{1}{p}, p})$  as initial value and repeat Steps 1 and 2 to obtain a unique mild solution  $U_2 \in Y(\kappa, 2\kappa)$  on  $[\kappa, 2\kappa]$  with initial data  $u_\kappa$ . One easily checks that letting  $U = U_1$  on  $[0, \kappa]$  and  $U = U_2$  on  $[\kappa, 2\kappa]$  defines a mild solution on  $[0, 2\kappa]$ . Iterating this finitely many times we obtain a mild solution  $U \in Y(0, T)$ .

*Step 5: Global uniqueness of mild solutions.*

To see that  $U$  is the unique mild solution in  $Y(0, T)$ , let  $V$  be another mild solution in  $Y(0, T)$ . Recall from Step 1 that we can find versions of  $U$  and  $V$  which also have paths in  $C([0, T]; X_{1-\frac{1}{p}, p})$ . It suffices to prove the uniqueness for these versions. Note that by the uniqueness on  $[0, \kappa]$  we have  $U|_{[0,\kappa]} = V|_{[0,\kappa]}$ . By the almost sure pathwise continuity of  $U$  and  $V$  with values in the space  $X_{1-\frac{1}{p}, p}$  we see that almost surely  $U(\kappa) = V(\kappa)$  in  $X_{1-\frac{1}{p}, p}$ . One easily checks that both  $U|_{[\kappa, 2\kappa]}$  and  $V|_{[\kappa, 2\kappa]}$  are mild solutions in  $Y(\kappa, 2\kappa)$  on the interval  $[\kappa, 2\kappa]$ . By uniqueness on  $[\kappa, 2\kappa]$  from Step 3a, we obtain that  $U|_{[\kappa, 2\kappa]} = V|_{[\kappa, 2\kappa]}$  in  $Y(\kappa, 2\kappa)$ . Proceeding in finitely many steps we obtain  $U = V$  in  $Y(0, T)$ .

*Step 6: The proof of part (ii).*

On  $[0, \kappa]$  it follows from Step 1 that

$$\begin{aligned} \|U\| &= \|L(U)\| \leq \|L(U) - L(0)\| + \|L(0)\| \\ &\leq (1 - \frac{1}{2}\theta)\|U\| + C(1 + \|u_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}). \end{aligned}$$

Since  $\theta \in (0, 1)$  we obtain

$$(4.2) \quad \|U\| \leq \frac{2C}{\theta} (1 + \|u_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}).$$

Next, observe that by Proposition 3.1, Theorems 3.3 and 3.5, Remark 4.1 (iv), and (HF) and (HB) one has

$$\begin{aligned} & \|U\|_{L^p(\Omega; C([0, \kappa]; X_{1-\frac{1}{p}, p}))} \\ &= \|L(U)\|_{L^p(\Omega; C([0, \kappa]; X_{1-\frac{1}{p}, p}))} \\ &\leq C\|u_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})} + K_p^* \|F(\cdot, U) + f\|_{Z_0} + K_p^\diamond \|B(\cdot, U) + b\|_{Z_1^{\frac{\gamma}{2}}} \\ &\leq C\|u_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})} + K_p^* C_{F,f} (1 + \|U\|_{Z_1}) + K_p^\diamond C_{B,b} (1 + \|U\|_{Z_1}). \end{aligned}$$

From (4.2) and the norm equivalence of  $\|\cdot\|$  on  $Z_1$  we obtain

$$(4.3) \quad \|U\|_{L^p(\Omega; C([0, \kappa]; X_{1-\frac{1}{p}, p}))} \leq \tilde{C} (1 + \|u_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})})$$

for some constant  $\tilde{C}$ . This proves the required estimates on  $[0, \kappa]$ . In particular, it follows from (4.3) that

$$(4.4) \quad \|U(\kappa)\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})} \leq \tilde{C} (1 + \|u_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}).$$

Using  $U(\kappa)$  as an initial values the same argument now gives the following estimates for  $U$  on  $[\kappa, 2\kappa]$ :

$$\begin{aligned} & \|U\|_{L_{\mathcal{F}}^p(\Omega; L^p(\kappa, 2\kappa; X_1))} \leq \frac{2C}{\theta} (1 + \|U(\kappa)\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}) \\ & \|U\|_{L^p(\Omega; C([\kappa, 2\kappa]; X_{1-\frac{1}{p}, p}))} \leq \tilde{C} (1 + \|U(\kappa)\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}). \end{aligned}$$

Combining this with (4.4) and iterating this finitely many times gives (2).

*Step 7:* The proof of part (iii).

First note that by Step 1,

$$\begin{aligned} \|U - V\|_{Z_1} &= \|L(U) - L(V) - Su_0 + Sv_0\|_{Z_1} \\ &\leq (1 - \frac{1}{2}\theta) \|U - V\|_{Z_1} + C \|u_0 - v_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}, \end{aligned}$$

where  $L = L_{u_0}$  is the operator from Step 1 with initial condition  $u_0$ .

Since  $\theta \in (0, 1)$  this implies

$$\|U - V\|_{Z_1} \leq \frac{2C}{\theta} \|u_0 - v_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}.$$

In the same way as for (4.3) one can prove that

$$\|U - V\|_{L^p(\Omega; C([0, \kappa]; X_{1-\frac{1}{p}, p}))} \leq \tilde{C} \|u_0 - v_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}.$$

Now one iterates the argument as in Steps 4 and 5.  $\square$

Theorem 4.5 can be seen as an extension of [7] to the borderline case. A maximal  $L^p$ -regularity result using real interpolation spaces instead of fractional domain spaces has been obtained in [6].



The first part of Hypothesis (HA)' implies that the operators  $-A(t, \omega)$  generate bounded analytic  $C_0$ -semigroups on  $X_0$  for which the estimate (3.1) holds uniformly in  $t$  and  $\omega$ .

Relatively continuous operators  $A$  have been introduced in [4] to study maximal  $L^p$ -regularity for deterministic problems. We consider a piecewise variant here, which seems to be new even in a deterministic setting. It seems that the results in [4] extend to this more general setting without difficulty.

**Definition 5.1.** *Let (H)' be satisfied. A process  $U : [0, T] \times \Omega \rightarrow X_0$  is called a strong solution of (SE') if it is strongly measurable and adapted, and*

(i) *almost surely,  $U \in L^2(0, T; X_1)$ ;*

(ii) *for all  $t \in [0, T]$ , almost surely the following identity holds in  $X_0$ :*

$$(5.1) \quad \begin{aligned} U(t) + \int_0^t A(s)U(s) ds &= u_0 + \int_0^t F(s, U(s)) + f(s) ds \\ &+ \int_0^t B(s, U(s)) + b(s) dW_H(s). \end{aligned}$$

As before, under (H)' all integrals are well defined, and again  $U$  has a pathwise continuous version for which, almost surely, the identity in (ii) holds for all  $t \in [0, T]$ .

**Theorem 5.2.** *Let (H)' be satisfied, let  $p \in [2, \infty)$ , and suppose that the operator family  $\mathcal{J}$  is  $R$ -bounded from  $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X_0))$  to  $L^p(\mathbb{R}_+ \times \Omega; X_0)$ . If the Lipschitz constants  $L_F$  and  $L_B$  satisfy*

$$K_p^* L_F + K_p^\diamond L_B < 1,$$

*then the assertions of Theorem 4.5 (i), (ii) and (iii) remain true for the problem (SE').*

*Proof.* As in the proof of Theorem 4.5 we may assume that  $K_p^* L_F + K_p^\diamond L_B = 1 - \theta$  with  $\theta \in (0, 1)$ .

Choose  $\delta > 0$  and  $\eta > 0$  such that for all  $1 \leq n \leq N$  and for all  $t, s \in [t_{n-1}, t_n]$ , for all  $x \in X_1$ ,

$$\|A(t)x - A(s)x\|_{X_0} \leq \frac{1}{2}\theta\|x\|_{X_1} + \eta\|x\|_{X_0} \text{ if } |t - s| < \delta.$$

Fix  $0 = s_0 < s_1 < \dots < s_M = T$  such that  $\{t_n : 0 \leq n \leq N\}$  is a subset of  $\{s_n : 0 \leq n \leq N\}$  and  $|s_m - s_{m-1}| < \delta$  for  $m = 1, \dots, M$ .

We first solve the problem on  $[0, s_1]$ . Let  $F_{A,0} : [0, s_1] \times \Omega \times X_1 \rightarrow X_0$  be defined by  $F_{A,0}(t, x) = F(t, x) - A(t)x + A(0)x$ . Then  $F_A$  satisfies (HF) with  $F$  replaced by  $F_{A,0}$ . Moreover,  $L_{F_{A,0}} \leq L_F + \frac{1}{2}\theta$  and  $\tilde{L}_{F_{A,0}} \leq \tilde{L}_F + C\eta$ , and therefore, the condition of Theorem 4.5 holds for the equation with  $F$  replaced by  $F_{A,0}$  and  $A$  replaced by  $A(0)$  with constant  $K_p^* L_{F_{A,0}} + K_p^\diamond L_B = 1 - \frac{1}{2}\theta$ . Therefore, Theorem 4.5 implies the existence of a unique strong solution  $U \in L^0_{\mathcal{F}}(\Omega; L^p(0, s_1; X_1))$ , i.e. almost surely, for all  $t \in [0, s_1]$  the following identity holds in  $X_0$ :

$$\begin{aligned} U(t) + \int_0^t A(0)U(s) ds &= u_0 + \int_0^t F_{A,0}(s, U(s)) + f(s) ds \\ &+ \int_0^t B(s, U(s)) + b(s) dW_H(s) \end{aligned}$$

and therefore also (5.1) holds on  $[0, s_1]$  almost surely. Moreover, the assertions of Theorem 4.5 (i), (ii) and (iii) hold on  $[0, s_1]$ .



- (d) for all  $R > 0$  a constant  $L_{F^{(2)},R}$  such that for all  $t \in [0, T]$ ,  $\omega \in \Omega$ , and  $x, y \in X_1$  satisfying  $\|x\|_{X_{1-\frac{1}{p},p}}, \|y\|_{X_{1-\frac{1}{p},p}} \leq R$ ,

$$\|F^{(2)}(t, \omega, x) - F^{(2)}(t, \omega, y)\|_{X_0} \leq L_{F^{(2)},R} \|x - y\|_{X_{1-\frac{1}{p},p}}$$

and there exists a constant  $C_{F^{(2)}}$  such that for all  $t \in [0, T]$ ,  $\omega \in \Omega$ ,

$$\|F^{(2)}(t, \omega, 0)\|_{X_0} \leq C_{F^{(2)}}.$$

(HB)<sub>loc</sub><sup>p</sup> The function  $b : [0, T] \times \Omega \rightarrow \gamma(H, X_{\frac{1}{2}})$  is adapted and strongly measurable and  $b \in L^2(0, T; \gamma(H, X_{\frac{1}{2}}))$  almost surely. The function  $B : [0, T] \times \Omega \times X_1 \rightarrow \gamma(H, X_{\frac{1}{2}})$  is given by  $B = B^{(1)} + B^{(2)}$ , where  $B^{(1)} : [0, T] \times \Omega \times X_1 \rightarrow \gamma(H, X_{\frac{1}{2}})$  and  $B^{(2)} : [0, T] \times \Omega \times X_{1-\frac{1}{p},p} \rightarrow \gamma(H, X_{\frac{1}{2}})$  are strongly measurable. The function  $B^{(1)}$  is  $\mathcal{F}$ -adapted and Lipschitz continuous, i.e., it satisfies (HB):

- (a) for all  $t \in [0, T]$  and  $x \in X_1$  the random variable  $\omega \mapsto B^{(1)}(t, \omega, x)$  is strongly  $\mathcal{F}_t$ -measurable;  
(b) there exist constants  $L_{B^{(1)}}$ ,  $\tilde{L}_{B^{(1)}}$ ,  $C_{B^{(1)}}$  such that for all  $t \in [0, T]$ ,  $\omega \in \Omega$ , and  $x, y \in X_1$ ,

$$\|B^{(1)}(t, \omega, x) - B^{(1)}(t, \omega, y)\|_{\gamma(H, X_{\frac{1}{2}})} \leq L_{B^{(1)}} \|x - y\|_{X_1} + \tilde{L}_{B^{(1)}} \|x - y\|_{X_0}$$

and

$$\|B^{(1)}(t, \omega, x)\|_{\gamma(H, X_{\frac{1}{2}})} \leq C_{B^{(1)}} (1 + \|x\|_{X_1}).$$

The function  $B^{(2)}$  is  $\mathcal{F}$ -adapted and locally Lipschitz continuous, i.e.,

- (c) for all  $t \in [0, T]$  and  $x \in X_{1-\frac{1}{p},p}$  the random variable  $\omega \mapsto B^{(2)}(t, \omega, x)$  is strongly  $\mathcal{F}_t$ -measurable;  
(d) for all  $R > 0$  a constant  $L_{B^{(2)},R}$  such that for all  $t \in [0, T]$ ,  $\omega \in \Omega$ , and  $x, y \in X_1$  satisfying  $\|x\|_{X_{1-\frac{1}{p},p}}, \|y\|_{X_{1-\frac{1}{p},p}} \leq R$ ,

$$\|B^{(2)}(t, \omega, x) - B^{(2)}(t, \omega, y)\|_{\gamma(H, X_{\frac{1}{2}})} \leq L_{B^{(2)},R} \|x - y\|_{X_{1-\frac{1}{p},p}}$$

and there exists a constant  $C_{B^{(2)}}$  such that for all  $t \in [0, T]$ ,  $\omega \in \Omega$ ,

$$\|B^{(2)}(t, \omega, 0)\|_{\gamma(H, X_{\frac{1}{2}})} \leq C_{B^{(2)}}.$$

Before we explain the definition of a local mild solution, we need to discuss some preliminaries on stopped stochastic convolutions. Let  $G : [0, T] \times \Omega \rightarrow \gamma(H, X_0)$  be an adapted process which satisfies  $G \in L^2(0, T; \gamma(H, X_0))$  almost surely. Let  $\tau$  be a stopping time with values in  $[0, T]$ . Define the  $X_0$ -valued processes  $I(G)$  by

$$I(G)(t) = \int_0^t S(t-s)G(s) dW_H(s).$$

As explained in [9] it is tempting to write

$$I(G)(t \wedge \tau) = \int_0^{t \wedge \tau} S(t \wedge \tau - s)G(s) dW_H(s).$$

This is meaningless, however, since the integrand in the right-hand expression is not adapted, and therefore the stochastic integral is not well defined. To remedy

this problem, following [9] we consider the process  $I_\tau(G)$  defined by

$$I_\tau(G)(t) = \int_0^t \mathbf{1}_{[0,\tau]}(s) S(t-s) G(s) dW_H(s) = S \diamond (\mathbf{1}_{[0,\tau]} G).$$

The following lemma can be proved as in [9, Lemma A.1].

**Lemma 5.3.** *Assume (HX). Let  $G : [0, T] \times \Omega \rightarrow \gamma(H, X_0)$  be an adapted process which satisfies  $G \in L^2(0, T; \gamma(H, X_0))$  almost surely. Let  $\tau$  be a stopping time with values in  $[0, T]$ . If the processes  $I(G)$  and  $I_\tau(G)$  have an  $X_0$ -valued continuous version, then almost surely,*

$$S(t - t \wedge \tau) I(G)(t) = I_\tau(G)(t), \quad t \in [0, T].$$

In particular, almost surely,

$$I(G)(t \wedge \tau) = I_\tau(G)(t \wedge \tau), \quad t \in [0, T].$$

Note that if  $G$  is only defined up to a stopping time  $\tau'$  with  $\tau \leq \tau'$  and  $\mathbf{1}_{[0,\tau]} G$  is in  $L^2(0, T; \gamma(H, X_0))$ , the above definition of  $I_\tau(G)$  is still meaningful. This is what we will use below.

*Remark 5.4.* If (HA) holds and  $G$  belongs to  $L^p(0, T; \gamma(H, X_0))$  almost surely for some  $p > 2$ , then Theorem 3.5 (combined with Remark 4.1 (iv)) shows that  $I(G)$  and  $I_\tau(G)$  are both pathwise continuous as  $X_{\frac{1}{2} - \frac{1}{p}, p}$ -valued processes, hence also as  $X_0$ -valued processes. For  $p = 2$ , pathwise continuity of  $I(G)$  and  $I_\tau(G)$  follows from [85, Theorem 1.1].

**Definition 5.5.** *Let  $p \in [2, \infty)$  and let  $(H)_{\text{loc}}^p$  be satisfied. Let  $\tau$  be a stopping time with values in  $[0, T]$ . A process  $U : [0, \tau] \times \Omega \rightarrow X_{1 - \frac{1}{p}, p}$  is called a local mild solution of (SE) if  $U$  is adapted and for each  $\omega \in \Omega$ ,  $t \mapsto U(t, \omega)$  is continuous in  $X_{1 - \frac{1}{p}, p}$  on the interval  $[0, \tau(\omega))$  and, for all  $n \geq 1$ ,*

- (i) almost surely,  $\mathbf{1}_{[0, \tau_n]} U \in L^2(0, T; X_1)$ ;
- (ii) almost surely, for all  $t \in [0, T]$ , the following identity holds in  $X_0$ :

$$\begin{aligned} U(t \wedge \tau_n) &= S(t \wedge \tau_n) u_0 + \int_0^{t \wedge \tau_n} S(t \wedge \tau_n - s) [F(s, U(s)) + f(s)] ds \\ &\quad + S \diamond (\mathbf{1}_{[0, \tau_n]} (B(\cdot, U) + b))(t \wedge \tau_n), \end{aligned}$$

where

$$\tau_n = \inf\{t \in [0, \tau) : \|U(t)\|_{X_{1 - \frac{1}{p}, p}} \geq n\}.$$

Note that

$$S \diamond (\mathbf{1}_{[0, \tau_n]} (B(\cdot, U) + b))(t \wedge \tau_n) = I_{\tau_n}(B(\cdot, U) + b)(t \wedge \tau_n).$$

The motivation for this expression has been explained in Lemma 5.3.

A process  $U : [0, \tau] \times \Omega \rightarrow X_{1 - \frac{1}{p}, p}$  is called a *maximal local mild solution* on  $[0, T]$  if it is a local mild solution and for every stopping time  $\tau'$  with values in  $[0, T]$  and every local mild solution  $V : [0, \tau'] \times \Omega \rightarrow X_{1 - \frac{1}{p}, p}$  one has  $\tau = \tau'$  almost surely and  $U = V$  in  $C([0, \tau]; X_{1 - \frac{1}{p}, p})$  almost surely. A process  $U : [0, T] \times \Omega \rightarrow X_{1 - \frac{1}{p}, p}$  is called a *global mild solution* if  $U$  is a local mild solution (with  $\tau = T$ ) and  $U \in L^2(0, T; X_1)$  almost surely. For such  $U$  one easily checks that part (ii) of Definition 4.3 holds.

In a similar way one can define local and global strong solutions. It is obvious from the proof of Proposition 4.4 that the notions of global strong solution and global mild solution are equivalent. Below we shall only consider local and global mild solutions.

The following theorem can be proved by following the lines of [6, 79] (see also [9] and [66, Theorem 8.1]).

**Theorem 5.6.** *Let  $(H)_{\text{loc}}^p$  be satisfied for  $p \in [2, \infty)$ , and suppose that the operator family  $\mathcal{J}$  is  $R$ -bounded from  $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X_0))$  to  $L^p(\mathbb{R}_+ \times \Omega; X_0)$ . If the Lipschitz constants  $L_F$  and  $L_B$  satisfy*

$$K_p^* L_F + K_p^\diamond L_B < 1,$$

then the following assertion holds:

- (i) *If  $u_0 \in L^0_{\mathcal{F}_0}(\Omega; X_{1-\frac{1}{p}, p})$ ,  $f \in L^0_{\mathcal{F}_0}(\Omega; L^p(0, T; X_0))$ , and  $b \in L^0_{\mathcal{F}_0}(\Omega; L^p(0, T; \gamma(H, X_{\frac{1}{2}})))$ , then the problem (SE) has a unique maximal local mild solution  $U$  in*

$$L^0_{\mathcal{F}}(\Omega; L^p(0, \tau; X_1)) \cap L^0_{\mathcal{F}}(\Omega; C([0, \tau]; X_{1-\frac{1}{p}, p})).$$

- (ii) *If, in addition to the assumptions in (i),  $F^{(2)}$  and  $B^{(2)}$  also satisfy the linear growth conditions*

$$\|F^{(2)}(t, \omega, x)\|_{X_0} \leq C_{F^{(2)}}(1 + \|x\|_{X_{1-\frac{1}{p}, p}}),$$

$$\|B^{(2)}(t, \omega, x)\|_{\gamma(H, X_{\frac{1}{2}})} \leq C_{B^{(2)}}(1 + \|x\|_{X_{1-\frac{1}{p}, p}}),$$

for some constants  $C_{F^{(2)}}$  and  $C_{B^{(2)}}$  independent of  $t \in [0, T]$ ,  $\omega \in \Omega$ , and  $x \in X_{1-\frac{1}{p}, p}$ , then the solution  $U$  in (i) is a global mild solution which belongs to

$$L^0_{\mathcal{F}}(\Omega; L^p(0, T; X_1)) \cap L^0_{\mathcal{F}}(\Omega; C([0, T]; X_{1-\frac{1}{p}, p})).$$

- (iii) *If, in addition to the assumptions of (i) and (ii), we have  $u_0 \in L^p_{\mathcal{F}_0}(\Omega; X_{1-\frac{1}{p}, p})$ ,  $f \in L^p_{\mathcal{F}}(\Omega; L^p(0, T; X_0))$ , and  $b \in L^p_{\mathcal{F}}(\Omega; L^p(0, T; \gamma(H, X_{\frac{1}{2}})))$ , then the global solution  $U$  in (ii) belongs to  $L^p_{\mathcal{F}}((0, T) \times \Omega; X_1) \cap L^p_{\mathcal{F}}(\Omega; C([0, T]; X_{1-\frac{1}{p}, p}))$  and satisfies*

$$\|U\|_{L^p((0, T) \times \Omega; X_1)} \leq C(1 + \|u_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}),$$

$$\|U\|_{L^p(\Omega; C([0, T]; X_{1-\frac{1}{p}, p}))} \leq C(1 + \|u_0\|_{L^p(\Omega; X_{1-\frac{1}{p}, p})}),$$

with constants  $C$  independent of  $u_0$ .

**5.3. The Hilbert space case.** For Hilbert spaces  $X_0$ , several of the constants in the estimates in Theorems 4.5 and 5.2 become explicit and we can give more precise conditions on the smallness of  $L_F$  and  $L_B$ . Below, we show that if  $A$  is self-adjoint and positive, then  $K_2^* \leq 1$  and  $K_2^\diamond \leq \frac{1}{\sqrt{2}}$  (these constants have been defined in the text preceding Theorem 4.5). Moreover, these estimates are optimal in the sense that the condition (5.3) below cannot be improved (see [78, Section 4.0] for the stochastic part; see also [13] for more information on the smallness condition for  $K_p^*$  and  $K_p^\diamond$  for  $p \neq 2$ ). As a consequence one obtains the following result, which is well known to experts (see [19, 78] for related results and [20] for applications to a class of SDPEs).

**Corollary 5.7.** *Let  $X_0$  and  $X_1$  be Hilbert spaces, and let  $A : [0, T] \times \Omega \rightarrow \mathcal{L}(X_1, X_0)$  be strongly measurable, adapted, self-adjoint, and piecewise relatively continuous uniformly on  $\Omega$ . Moreover, assume that there is a constant  $\delta > 0$  such that*

$$\|e^{sA(t, \omega)}\| \leq e^{-\delta s}, \quad t \in [0, T], \quad \omega \in \Omega.$$

Assume (HF), (HB) and (Hu<sub>0</sub>). The assertions of Theorem 5.2 hold whenever

$$(5.3) \quad L_F + \frac{L_B}{\sqrt{2}} < 1.$$

A similar consequence of Theorem 5.6 can be formulated in the Hilbert space setting.

*Proof.* The result follows at once from Theorem 5.2 once we show that  $K_2^* \leq 1$  and  $K_2^\diamond \leq \frac{1}{\sqrt{2}}$ . Here it is important to endow  $X_{\frac{1}{2}}$  with the norm

$$(5.4) \quad \|x\|_{\frac{1}{2}} := \|A^{\frac{1}{2}}x\|$$

(cf. the discussion below (3.2)). By the invertibility of  $A$  and the equivalence of norms (3.2), (5.4) indeed defines an equivalent norm on  $X_{\frac{1}{2}}$ . If what follows, we understand  $K_2^*$  and  $K_2^\diamond$  as the operator norms as defined in Section 4, with  $X_{\frac{1}{2}}$  normed by (5.4).

We first show that  $K_2^* \leq 1$ . Using the spectral theorem one can see that for all  $s \in \mathbb{R}$ , one has

$$(5.5) \quad \|A(is + A)^{-1}\| \leq 1$$

As direct proof is obtained as follows. For  $x \in X_0$  with  $\|x\| \leq 1$  and  $s \in \mathbb{R}$  one has

$$\begin{aligned} \|A(is + A)^{-1}x\|^2 &= \langle A^2(-is + A)^{-1}(is + A)^{-1}x, x \rangle \\ &= \langle A^2(s^2 + A^2)^{-1}x, x \rangle = \langle A^2(t + A^2)^{-1}x, x \rangle =: f(t), \end{aligned}$$

where  $t = s^2$ . Then  $f(0) = 1$  and, for  $t > 0$ ,

$$f'(t) = -\langle A^2(t + A^2)^{-2}x, x \rangle = -\|A(t + A^2)^{-1}x\|^2 \leq 0,$$

and therefore  $f(t) \leq 1$  as claimed.

By (5.5) and Plancherel's theorem, for any  $g \in L^2(\mathbb{R}_+; X_0)$  one has that

$$\|AS * g\|_{L^2(\mathbb{R}_+; X_0)}^2 = \int_{\mathbb{R}} \|A(is + A)^{-1}\hat{g}(s)\|_{X_0}^2 ds \leq \int_{\mathbb{R}} \|\hat{g}(s)\|_{X_0}^2 ds = \|g\|_{L^2(\mathbb{R}_+; X_0)}^2,$$

and hence  $K_2^* \leq 1$ .

Next we show that  $K_2^\diamond \leq \frac{1}{\sqrt{2}}$  (cf. [19, Section 6.3.2]). By standard arguments involving the essentially separable-valuedness of strongly measurable mappings (cf. [64]) there is no loss of generality in assuming that  $H$  is separable. Let  $(h_n)_{n \geq 1}$  be an orthonormal basis of  $H$ . Let  $\mathcal{L}_2(H, X_{\frac{1}{2}})$  denote the space of Hilbert-Schmidt operators (which is canonically isometric to  $\gamma(H, X_{\frac{1}{2}})$ ). By the Itô isometry, for all  $G \in L^2(\mathbb{R}_+ \times \Omega; \mathcal{L}_2(H, X_{\frac{1}{2}}))$  we have

$$\begin{aligned} \|A^{\frac{1}{2}}S \diamond G\|_{L^2(\mathbb{R}_+ \times \Omega; X_{\frac{1}{2}})}^2 &= \int_0^\infty \int_0^t \sum_{n \geq 1} \mathbb{E} \|AS(t-s)G(s)h_n\|^2 ds dt \\ &\leq \int_0^\infty \int_0^\infty \sum_{n \geq 1} \mathbb{E} \|AS(t)G(s)h_n\|^2 dt ds \end{aligned}$$



- (iii) There is a constant  $M_2 \geq 0$  and an angle  $\vartheta \in [0, \frac{1}{2}\pi)$  such that for all  $t \in [0, T]$ ,  $\omega \in \Omega$ ,  $x \in \mathbb{R}^d$ , and  $\xi \in \mathbb{R}^d$  with  $|\xi| = 1$  we have

$$\sigma(\mathcal{A}_\pi(t, \omega, x, \xi)) \subseteq \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leq \vartheta\}$$

and

$$\|\mathcal{A}_\pi(t, \omega, x, \xi)^{-1}\|_{\mathcal{L}(\mathbb{C}^N)} \leq M_2.$$

Let  $A_q(t, \omega)$  denote the realization of  $\mathcal{A}(t, \omega, \cdot)$  in  $L^q(\mathbb{R}^d; \mathbb{C}^N)$  with domain

$$D(A(t, \omega)) = H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N).$$

By [28, Theorem 6.1], applied pointwise on  $\Omega$ , one has the following powerful result for the  $H^\infty$ -calculus of  $A$ .

**Proposition 6.1** ([28]). *Let Hypothesis (Ha) be satisfied. For all  $q \in (1, \infty)$  and  $\sigma \in (\vartheta, \frac{1}{2}\pi)$  there exist constants  $w \geq 0$  and  $C \geq 1$ , depending only on  $q, \sigma, \vartheta, M_1, M_2$ , such that for all  $\omega \in \Omega$  and  $t \in [0, T]$  the operator  $A_q(\omega, t) + w$  has a bounded  $H^\infty(\Sigma_\sigma)$ -calculus on  $L^q(\mathbb{R}^d; \mathbb{C}^N)$  with boundedness constant at most  $C$ .*

This result actually holds with  $\vartheta \in [0, \pi)$ , provided one extends the definition of bounded  $H^\infty$ -calculi accordingly (replacing negative generators of analytic semi-groups by general sectorial operators), but we shall not need it in this generality.

## 6.2. Hypotheses on the functions $f, f^0, b, b^0$ , and the initial value $u_0$ .

- (Hf) The function  $f^0 : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow L^q(\mathbb{R}^d; \mathbb{C}^N)$  is  $\mathcal{P} \times \mathcal{B}_{\mathbb{R}^d}$ -measurable and satisfies  $f^0 \in L^1(0, T; L^q(\mathbb{R}^d; \mathbb{C}^N))$  almost surely. The function  $f : [0, T] \times \Omega \times \mathbb{R}^d \times H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N) \rightarrow L^q(\mathbb{R}^d; \mathbb{C}^N)$  is  $\mathcal{P} \times \mathcal{B}_{\mathbb{R}^d} \times \mathcal{B}(H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N))$ -measurable. There exist constants  $\alpha_f \in [0, 1]$ ,  $L_f \geq 0$ ,  $L_{f, \alpha_f} \geq 0$ ,  $C_f \geq 0$  such that for all  $u, v \in H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N)$ ,  $t \in [0, T]$ , and  $\omega \in \Omega$  one has

$$\begin{aligned} & \|f(t, \omega, \cdot, u) - f(t, \omega, \cdot, v)\|_{L^q(\mathbb{R}^d; \mathbb{C}^N)} \\ & \leq L_f \|u - v\|_{H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N)} + L_{f, \alpha_f} \|u - v\|_{H^{2m - \alpha_f, q}(\mathbb{R}^d; \mathbb{C}^N)} \end{aligned}$$

and

$$\|f(t, \omega, u)\|_{L^q(\mathbb{R}^d; \mathbb{C}^N)} \leq C_f(1 + \|u\|_{H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N)}).$$

- (Hb) The functions  $b_i^0 : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow H^{m, q}(\mathbb{R}^d; \mathbb{C}^N)$  are  $\mathcal{P} \times \mathcal{B}_{\mathbb{R}^d}$ -measurable and satisfy  $b^0 \in L^1(0, T; H^{m, q}(\mathbb{R}^d; \ell^2(\mathbb{C}^N)))$  almost surely. The functions  $b_i : [0, T] \times \Omega \times \mathbb{R}^d \times H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N) \rightarrow H^{m, q}(\mathbb{R}^d; \mathbb{C}^N)$  are  $\mathcal{P} \times \mathcal{B}_{\mathbb{R}^d} \times \mathcal{B}(H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N))$ -measurable. There exist constants  $\alpha_b \in [0, 1]$ ,  $L_b \geq 0$ ,  $L_{b, \alpha_b} \geq 0$  and  $C_b$  such that for all  $u, v \in H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N)$ ,  $t \in [0, T]$ , and  $\omega \in \Omega$  one has

$$\begin{aligned} & \|b(t, \omega, \cdot, u) - b(t, \omega, \cdot, v)\|_{H^{m, q}(\mathbb{R}^d; \ell^2(\mathbb{C}^N))} \\ & \leq L_b \|u - v\|_{H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N)} + L_{b, \alpha_b} \|u - v\|_{H^{2m - \alpha_b, q}(\mathbb{R}^d; \mathbb{C}^N)} \end{aligned}$$

and

$$\|b(t, \omega, u)\|_{H^{m, q}(\mathbb{R}^d; \ell^2(\mathbb{C}^N))} \leq C_b(1 + \|u\|_{H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N)}).$$

- (Hu<sub>0</sub>) The initial value  $u_0 : \Omega \rightarrow L^q(\mathbb{R}^d; \mathbb{C}^N)$  is  $\mathcal{F}_0$ -measurable.

**6.3. Main result.** We begin by defining the notion of a strong solutions to the SPDE (6.1). We fix exponents  $p, q \in [2, \infty)$  and assume that (Ha), (Hf), (Hb), (Hu<sub>0</sub>) are satisfied. As in Section 4 it can be shown that a strong solution with paths in  $L^p(0, T; H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N))$  is also mild and weak solution (cf. Proposition 4.4 and the references given there).

**Definition 6.2.** *A progressively measurable process  $u \in L^0(\Omega; L^p(0, T; H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N)))$  is called a strong solution to (6.1) if for almost all  $(t, \omega) \in [0, T] \times \Omega$ ,*

$$\begin{aligned} u(t, \cdot) + \int_0^t \mathcal{A}(s, \cdot, D)u(s, \cdot) ds &= u_0(\cdot) + \int_0^t f(s, \cdot, u(s, \cdot)) + f^0(s, \cdot) ds \\ &+ \sum_{i \geq 1} \int_0^t b_i(s, \cdot, u(s, \cdot)) + b_i^0(s, \cdot) dw_i(s). \end{aligned}$$

The integral with respect to time is well defined as a Bochner integral in the space  $L^q(\mathbb{R}^d; \mathbb{C}^N)$ . By (2.5) and the remark following it, the stochastic integrals are well defined in the space  $H^{m, q}(\mathbb{R}^d; \mathbb{C}^N)$ . Indeed, by (Hb) and the isomorphism (2.3) one has

$$\begin{aligned} \|b(s, \cdot, u(s, \cdot))\|_{\gamma(\ell^2, H^{m, q}(\mathbb{R}^d; \mathbb{C}^N))} &\approx_q \|b(s, \cdot, u(s, \cdot))\|_{H^{m, q}(\mathbb{R}^d; \ell^2(\mathbb{C}^N))} \\ &\leq C_b(1 + \|u(s, \cdot)\|_{H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N)}). \end{aligned}$$

By the assumptions on  $u$ , the  $L^2(0, T)$ -norm of the right-hand side is finite almost surely.

As a consequence of Theorem 5.2 one has the following well-posedness result for the SPDE (6.1).

**Theorem 6.3.** *Let  $q \in [2, \infty)$  and  $p \in (2, \infty)$ , where  $p = 2$  is also allowed if  $q = 2$ . Assume (Ha), (Hf), (Hb), (Hu<sub>0</sub>), and suppose that  $f^0 \in L^p_{\mathcal{F}}(\Omega; L^p(0, T; L^q(\mathbb{R}^d; \mathbb{C}^N)))$  and  $b^0 \in L^p_{\mathcal{F}}(\Omega; L^p(0, T; H^{m, q}(\mathbb{R}^d; \ell^2(\mathbb{C}^N))))$ . Provided  $L_f$  and  $L_b$  are small enough, the following assertions hold:*

- (i) *If  $u_0 \in L^0_{\mathcal{F}_0}(\Omega; B_{q, p}^{2m(1-\frac{1}{p})}(\mathbb{R}^d; \mathbb{C}^N))$ , then the problem (6.1) has a unique solution  $u \in L^0_{\mathcal{F}}(\Omega; L^p(0, T; H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N)))$ . Moreover,  $u$  has a version with trajectories in  $C([0, T]; B_{q, p}^{2m(1-\frac{1}{p})}(\mathbb{R}^d; \mathbb{C}^N))$ .*
- (ii) *If  $u_0 \in L^p_{\mathcal{F}_0}(\Omega; B_{q, p}^{2m(1-\frac{1}{p})}(\mathbb{R}^d; \mathbb{C}^N))$ , then the solution  $u$  given by part (i) satisfies*

$$\begin{aligned} \|u\|_{L^p((0, T) \times \Omega; H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N))} &\leq C(1 + \|u_0\|_{L^p(\Omega; B_{q, p}^{2m(1-\frac{1}{p})}(\mathbb{R}^d; \mathbb{C}^N))}) \\ \|u\|_{L^p(\Omega; C([0, T]; B_{q, p}^{2m(1-\frac{1}{p})}(\mathbb{R}^d; \mathbb{C}^N))} &\leq C(1 + \|u_0\|_{L^p(\Omega; B_{q, p}^{2m(1-\frac{1}{p})}(\mathbb{R}^d; \mathbb{C}^N))}), \end{aligned}$$

with constants  $C$  independent of  $u_0$ .

- (iii) *For all  $u_0, v_0 \in L^p_{\mathcal{F}_0}(\Omega; B_{q, p}^{2m(1-\frac{1}{p})}(\mathbb{R}^d; \mathbb{C}^N))$ , the corresponding solutions  $u, v$  satisfy*

$$\begin{aligned} \|u - v\|_{L^p((0, T) \times \Omega; H^{2m, q}(\mathbb{R}^d; \mathbb{C}^N))} &\leq C\|u_0 - v_0\|_{L^p(\Omega; B_{q, p}^{2m(1-\frac{1}{p})}(\mathbb{R}^d; \mathbb{C}^N))}, \\ \|u - v\|_{L^p(\Omega; C([0, T]; B_{q, p}^{2m(1-\frac{1}{p})}(\mathbb{R}^d; \mathbb{C}^N))} &\leq C\|u_0 - v_0\|_{L^p(\Omega; B_{q, p}^{2m(1-\frac{1}{p})}(\mathbb{R}^d; \mathbb{C}^N))}, \end{aligned}$$

with constants  $C$  independent of  $u_0$  and  $v_0$ .

*Proof.* It suffices to check the conditions of Theorem 5.2 with  $X_0 = L^q(\mathbb{R}^d; \mathbb{C}^N)$  and  $X_1 = H^{2m,q}(\mathbb{R}^d; \mathbb{C}^N)$ . These spaces satisfy Hypothesis (HX), Hypothesis (HA)' holds by Proposition 6.1 and the assumption that  $\vartheta < \frac{1}{2}\pi$ , and Hypothesis (Hu<sub>0</sub>) holds by the assumption on  $u_0$ . The family  $\mathcal{J}$  is  $R$ -bounded from  $\mathcal{L}(L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X_0)))$  to  $L^p(\mathbb{R}_+ \times \Omega; X_0)$  by Theorem 2.5.

Recall from [83, Theorems 2.4.2, 2.4.7 and 2.5.6] that

$$(6.2) \quad X_{\frac{1}{2}} = H^{m,q}(\mathbb{R}^d; \mathbb{C}^N) \quad \text{and} \quad X_{1-\frac{1}{p},p} = B_{q,p}^{2m(1-\frac{1}{p})}(\mathbb{R}^d; \mathbb{C}^N).$$

Let  $F : [0, T] \times \Omega \times X_1 \rightarrow X_0$  be defined by  $F(t, \omega, u) = f(t, \omega, \cdot, u)$ . The additional additive term can be defined in a similar way. Then the equivalent version of (HF) discussed in Remark 4.1 is satisfied with  $\alpha_F = \alpha_f$ ,  $L'_F = L_f$ ,  $\tilde{L}'_F = L_{f,\alpha_f}$  and  $C_F = C_f$ . Let  $H = \ell^2$  and let  $B : [0, T] \times \Omega \times X_1 \rightarrow \gamma(H, X_{\frac{1}{2}})$  be defined by  $B(t, \omega, u)e_i = b_i(t, \omega, \cdot, u)$ . The additional additive term can be defined in a similar way. Then the equivalent version of (HB) discussed in Remark 4.1 is satisfied with  $\alpha_B = \alpha_b$ ,  $L'_B = L_b$ ,  $\tilde{L}'_B = L_{b,\alpha_b}$  and  $C_B = C_b$ .

In this way, the equation (6.1) can be written as (SE'), where the unknown processes  $u : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{C}^N$  and  $U : [0, T] \times \Omega \rightarrow X_0$  are identified through  $U(t, \omega)(x) = u(t, \omega, x)$ . The result then follows from Theorem 5.2 and (6.2).  $\square$

*Remark 6.4.* Let  $n \in \mathbb{Z}$ . If  $a_\alpha \in BUC^{|n|}(\mathbb{R}^d; \mathbb{C}^N \times \mathbb{C}^N)$  one can transfer the result of Proposition 6.1 to the realization of  $\mathcal{A}(t, \omega, \cdot)$  in  $H^{n,q}(\mathbb{R}^d; \mathbb{C}^N)$  with domain

$$D(A_{n,q}(t, \omega)) = H^{n+2m,q}(\mathbb{R}^d; \mathbb{C}^N).$$

We refer to [47, Lemma 5.2] for details. Using this fact, under suitably reformulated assumptions on  $f$ ,  $f^0$ ,  $b$ ,  $b^0$  and  $u_0$  one can obtain a version of Theorem 6.3 with an additional regularity parameter  $n \in \mathbb{Z}$ . It is even possible to consider a real parameter  $n$ , but in that case one needs additional smoothness on  $a$  (see [83, Corollary 2.8.2]).

**6.4. Discussion.** In this subsection we compare the above result Theorem 6.3 with available results in the literature.

The case  $m = 1$  and  $N = 1$  of Theorem 6.3 has some overlap with [47, Theorem 5.1] due to Krylov. Theorem 6.3 improves on [47, Theorem 5.1] in various respects.

- (i) Our approach covers SPDEs governed by  $N$ -dimensional systems of elliptic operators of order  $2m$  for any  $m \geq 1$ .

Even for  $m = 1$  and  $N = 1$ , there are new features in our approach:

- (ii) In our setting, the highest order coefficients  $a_\alpha$  are only assumed to be bounded and uniformly continuous in the space variable, whereas in [47, Theorem 5.1] it is assumed that they are Hölder continuous in the space variable. Our continuity assumptions can be further weakened to VMO assumptions (cf. [29] for the second order case). Recently, in [43] Krylov's  $L^p$ -approach has been extended to prove results for continuous coefficients as well.
- (iii) In our approach, the parameters  $p$  and  $q$  can be chosen independently of each other. In [47, Theorem 5.1], only the case  $p = q$  is considered, in [48] an extension to the case  $p \geq q \geq 2$  was obtained. We do not need such an assumption.



where  $D_i$  denotes the  $i$ -th partial derivative, and

$$\mathcal{C}(x, D) = \sum_{i=1}^d c_i(x) D_i + c_0(x).$$

**7.1. Assumptions on the coefficients  $a_{ij}$ ,  $a_i$ ,  $c_i$ .** Essentially, the assumptions on  $a_{ij}$  and  $a_i$  correspond to a special case of an example in [21] and [40].

(Ha) The coefficients  $a_{ij}$ ,  $a_i$ ,  $c_i$  are real-valued and satisfy:

(i) There is a constant  $\rho \in (0, 1]$  such that

$$a_{ij} \in C^\rho(\overline{\mathcal{O}}) \text{ for all } 1 \leq i, j \leq d.$$

Furthermore,

$$a_i \in C(\overline{\mathcal{O}}) \text{ for all } 0 \leq i \leq n,$$

$$c_i \in C^1(\overline{\mathcal{O}}) \text{ for all } 0 \leq i \leq d.$$

(ii) The matrices  $(a_{ij}(x))$  are symmetric and there is a constant  $\kappa > 0$  such that for all  $x \in \mathcal{O}$  and  $\xi \in \mathbb{R}^d$  one has

$$\sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq \kappa |\xi|^2.$$

(iii) For all  $x \in \Gamma_0$  we have  $c_0(x) = 1$  and  $c_1(x) = c_2(x) = \dots = c_d(x) = 0$ . There is a constant  $\kappa' > 0$  such that for all  $x \in \Gamma_1$  we have

$$\sum_{i,j=1}^d c_i(x) n_i(x) \geq \kappa'.$$

We denote by  $A_q$  be the realization of  $\mathcal{A}(\cdot)$  in  $L^q(\mathcal{O})$  with domain

$$\mathcal{D}(A(t, \omega)) = H_C^{2,q}(\mathcal{O}) := \{u \in H^{2,q}(\mathcal{O}) : \mathcal{C}(x, D)u = 0\}.$$

One has the following result for the  $H^\infty$ -calculus of  $A_q$  (see [21] and [40]).

**Proposition 7.1.** *Assume that (Ha) is satisfied. For all  $q \in (1, \infty)$  there exist constants  $w \geq 0$  and  $\sigma \in [0, \frac{1}{2}\pi)$  such that  $A_q + w$  has a bounded  $H^\infty(\Sigma_\sigma)$ -calculus on  $L^q(\mathcal{O})$ .*

**7.2. Hypotheses on the functions  $f$ ,  $f^0$ ,  $b$ ,  $b^0$ , and the initial value  $u_0$ .**

(Hf) The function  $f^0 : [0, T] \times \Omega \times \mathcal{O} \rightarrow L^q(\mathcal{O})$  is  $\mathcal{P} \times \mathcal{B}_{\mathcal{O}}$ -measurable and satisfies  $f^0 \in L^1(0, T; L^q(\mathcal{O}))$  almost surely. The function  $f : [0, T] \times \Omega \times \mathcal{O} \times H_C^{2,q}(\mathcal{O}) \rightarrow L^q(\mathcal{O})$  is  $\mathcal{P} \times \mathcal{B}_{\mathcal{O}} \times \mathcal{B}(H^{2,q}(\mathcal{O}))$ -measurable and there exist constants  $\alpha_f \in [0, 1)$ ,  $L_f \geq 0$ ,  $L_{f,\alpha_f} \geq 0$ , and  $C_f \geq 0$  such that for all  $u, v \in H_C^{2,q}(\mathcal{O})$ ,  $t \in [0, T]$ , and  $\omega \in \Omega$  one has

$$\begin{aligned} & \|f(t, \omega, \cdot, u) - f(t, \omega, \cdot, v)\|_{L^q(\mathcal{O})} \\ & \leq L_f \|u - v\|_{H^{2,q}(\mathcal{O})} + L_{f,\alpha_f} \|u - v\|_{H^{2-\alpha_f,q}(\mathcal{O})}, \end{aligned}$$

and

$$\|f(t, \omega, u)\|_{L^q(\mathcal{O})} \leq C_f (1 + \|u\|_{H^{2,q}(\mathcal{O})}).$$

(Hb) The functions  $b_i^0 : [0, T] \times \Omega \times \mathcal{O} \rightarrow L^q(\mathcal{O})$  are  $\mathcal{P} \times \mathcal{B}_{\mathcal{O}}$ -measurable and satisfy  $b^0 \in L^1(0, T; H^{1,q}(\mathcal{O}; \ell^2))$  almost surely. The functions  $b_i : [0, T] \times \Omega \times \mathcal{O} \times H_C^{2,q}(\mathcal{O}) \rightarrow H^{1,q}(\mathcal{O})$  are  $\mathcal{P} \times \mathcal{B}_{\mathcal{O}} \times \mathcal{B}(H_C^{2,q}(\mathcal{O}))$ -measurable and there exist constants  $\alpha_b \in [0, 1)$ ,  $L_{b,1} \geq 0$ ,  $L_{b,\alpha_b} \geq 0$ , and  $C_b$  such that for all  $u, v \in H_C^{2,q}(\mathcal{O})$ ,  $t \in [0, T]$ , and  $\omega \in \Omega$  one has

$$\begin{aligned} & \|b(t, \omega, \cdot, u) - b(t, \omega, \cdot, v)\|_{H_C^{1,q}(\mathcal{O})} \\ & \leq L_b \|u - v\|_{H^{2,q}(\mathcal{O})} + L_{b,\alpha_b} \|u - v\|_{H^{2-\alpha_b,q}(\mathcal{O})} \end{aligned}$$

and

$$\|b(t, \omega, u)\|_{H_C^{1,q}(\mathcal{O}; \ell^2)} \leq C_b(1 + \|u\|_{H^{2,q}(\mathcal{O})}).$$

(Hu<sub>0</sub>) The initial value  $u_0 : \Omega \rightarrow L^q(\mathcal{O})$  is  $\mathcal{F}_0$ -measurable.

**7.3. Main result.** We let  $p, q \in [2, \infty)$  and assume that (Ha), (Hf) (Hb), (Hu<sub>0</sub>) are satisfied.

**Definition 7.2.** A progressively measurable process  $u \in L^0(\Omega; L^p(0, T; H^{2,q}(\mathcal{O})))$  is called a solution to (7.1) if, for almost all  $(t, \omega) \in [0, T] \times \Omega$ ,

$$\begin{aligned} u(t, \cdot) + \int_0^t \mathcal{A}(\cdot, D)u(s, \cdot) ds &= u_0(\cdot) + \int_0^t f(s, \cdot, u(s, \cdot)) + f^0(s, \cdot) ds \\ &+ \sum_{i \geq 1} \int_0^t b_i(s, \cdot, u(s, \cdot)) + b_i^0(s, \cdot) dw_i(s). \end{aligned}$$

Arguing as in the previous section, we see that the integral with respect to time is well defined as a Bochner integral in the space  $L^q(\mathcal{O})$  and the stochastic integrals are well defined in  $H_C^{1,q}(\mathcal{O})$ .

Following [1], we define the following Besov and Bessel potential spaces with boundary conditions. For  $p \in (1, \infty)$  and  $q \in (1, \infty)$ , and  $S_q^s \in \{B_{q,p}^s, H_q^s\}$  let

$$S_{q,\mathcal{C}}^s(\mathcal{O}) = \{u \in S_q^s(\mathcal{O}) : \mathcal{C}u = 0\} \quad 1 + \frac{1}{q} < s \leq 2.$$

For  $p \in (1, \infty)$  and  $q \in (1, \infty)$ , and  $S_q^s \in \{B_{q,p}^s, H_q^s\}$  let

$$S_{q,\mathcal{C}}^s(\mathcal{O}) = \{u \in S_q^s(\mathcal{O}) : \text{Tr}(u) = 0 \text{ on } \Gamma_0\} \quad \frac{1}{q} < s < 1 + \frac{1}{q}.$$

Below we use the following well-known result:

$$X_{\frac{1}{2}} = H_C^{1,q}(\mathcal{O}) \quad \text{and} \quad X_{1-\frac{1}{p},p} = B_{q,p,\mathcal{C}}^{2-\frac{2}{p}}(\mathcal{O}),$$

Indeed, since  $A$  has a bounded  $H^\infty$ -calculus of angle  $< \frac{1}{2}\pi$ , it has bounded imaginary powers and therefore, by [84, Theorem 1.15.3],  $X_{\frac{1}{2}} = [X_0, X_1]_{\frac{1}{2}}$  with equivalent norms. Now by Theorem 5.2 and Remark 5.3 (c) in [1] one obtains  $X_{\frac{1}{2}} = H_C^{1,q}(\mathcal{O})$  with equivalent norms. Similarly, if  $2 - \frac{2}{p} \notin \{\frac{1}{q}, 1 + \frac{1}{q}\}$  then

$$X_{1-\frac{1}{p},p} = (X_0, X_1)_{1-\frac{1}{p},p} = B_{q,p,\mathcal{C}}^{2-\frac{2}{p}}(\mathcal{O}).$$

Note that in the case that  $\Gamma_0 = \emptyset$ , one has  $H_C^{1,q}(\mathcal{O}) = H^{1,q}(\mathcal{O})$  for all  $q \in (1, \infty)$  with equivalent norms.

As a consequence of Theorem 4.5 we obtain the following well-posedness result for the SPDE (6.1).

**Theorem 7.3.** *Let  $q \in [2, \infty)$  and  $p \in (2, \infty)$ , where  $p = 2$  is also allowed if  $q = 2$ . Assume that  $\frac{2}{p} + \frac{1}{q} \neq 1$ . Assume that (Ha), (Hf), (Hb), (Hu<sub>0</sub>) are satisfied and suppose that  $f^0 \in L^p_{\mathcal{F}}(\Omega; L^p(0, T; L^q(\mathcal{O})))$  and  $b^0 \in L^p_{\mathcal{F}}(\Omega; L^p(0, T; H^{1,q}(\mathcal{O}; \ell^2)))$ . Provided  $L_f$  and  $L_b$  are small enough, the following assertions hold:*

- (i) *If  $u_0 \in L^0_{\mathcal{F}_0}(\Omega; B^{2-\frac{2}{p}}_{q,p,C}(\mathcal{O}))$ , then the problem (6.1) has a unique solution  $u \in L^0_{\mathcal{F}}(\Omega; L^p(0, T; H^{2,q}_C(\mathcal{O})))$ . Moreover,  $u$  has a version with trajectories in the space  $C([0, T]; B^{2-\frac{2}{p}}_{q,p,C}(\mathcal{O}))$ .*
- (ii) *If  $u_0 \in L^p_{\mathcal{F}_0}(\Omega; B^{2-\frac{2}{p}}_{q,p,C}(\mathcal{O}))$ , then the solution  $u$  given by part (i) satisfies*

$$\begin{aligned} \|u\|_{L^p((0,T) \times \Omega; H^{2,q}_C(\mathcal{O}))} &\leq C(1 + \|u_0\|_{L^p(\Omega; B^{2-\frac{2}{p}}_{q,p,C}(\mathcal{O}))}) \\ \|u\|_{L^p(\Omega; C([0,T]; B^{2-\frac{2}{p}}_{q,p,C}(\mathcal{O})))} &\leq C(1 + \|u_0\|_{L^p(\Omega; B^{2-\frac{2}{p}}_{q,p,C}(\mathcal{O}))}), \end{aligned}$$

with constants  $C$  independent of  $u_0$ .

- (iii) *For all  $u_0, v_0 \in L^p_{\mathcal{F}_0}(\Omega; B^{2-\frac{2}{p}}_{q,p,C}(\mathcal{O}))$ , the corresponding solutions  $u, v$  satisfy*

$$\begin{aligned} \|u - v\|_{L^p((0,T) \times \Omega; H^{2,q}_C(\mathcal{O}))} &\leq C\|u_0 - v_0\|_{L^p(\Omega; B^{2-\frac{2}{p}}_{q,p,C}(\mathcal{O}))}, \\ \|u - v\|_{L^p(\Omega; C([0,T]; B^{2-\frac{2}{p}}_{q,p,C}(\mathcal{O})))} &\leq C\|u_0 - v_0\|_{L^p(\Omega; B^{2-\frac{2}{p}}_{q,p,C}(\mathcal{O}))}, \end{aligned}$$

with constants  $C$  independent of  $u_0$  and  $v_0$ .

*Proof.* We check the conditions of Theorem 5.2 with  $X_0 = L^q(\mathcal{O})$  and  $X_1 = H^{2,q}_C(\mathcal{O})$ .

As in the proof of Theorem 6.3, the verification of the Hypotheses (HX), (HA), (Hu<sub>0</sub>), as well as the  $R$ -boundedness of  $\mathcal{J}$  is immediate.

Let  $F : [0, T] \times \Omega \times X_1 \rightarrow X_0$  be defined by  $F(t, \omega, u) = f(t, \omega, \cdot, u)$ . The additional term can be defined in a similar way. Then the equivalent version of (HF) discussed in Remark 4.1 is satisfied with  $\alpha_F = \alpha_f$ ,  $L'_F = L_f$ ,  $\tilde{L}'_F = L_{f, \alpha_f}$  and  $C_F = C_f$ . Let  $H = \ell^2$  and let  $B : [0, T] \times \Omega \times X_1 \rightarrow \gamma(H, X_{\frac{1}{2}})$  be defined by  $B(t, \omega, u)e_i = b_i(t, \omega, \cdot, u)$ . The additional term can be defined in a similar way. Then (HB) (see Remark 4.1) is satisfied with  $\alpha_B = \alpha_b$ ,  $L'_B = L_b$ ,  $\tilde{L}'_B = L_{b, \alpha_b}$  and  $C_B = C_b$ .

In this way, the equation (6.1) can be written as (SE'), where the unknown processes  $u : [0, T] \times \Omega \times \mathcal{O} \rightarrow \mathbb{R}$  and  $U : [0, T] \times \Omega \rightarrow X_0$  are identified through  $U(t, \omega)(x) = u(t, \omega, x)$ . The result now follows from Theorem 5.2 and (6.2) and the assumptions on  $p$  and  $q$ .  $\square$

*Remark 7.4.* Under additional continuity assumptions on the coefficients  $a_{ij}$ , the same methods one can be used to handle the case where  $\mathcal{A}$  depends on time and  $\Omega$ .

**7.4. Discussion.** In case of Dirichlet boundary conditions, related results for weighted half-spaces and bounded domains with weights have been obtained by Kim and Krylov (see [43] and references therein) using Krylov's  $L^p$ -approach. The weighted approach started with the  $L^2$ -theory of Krylov [45]. The advantage of using weights is that no additional compatibility conditions on the noise are required in this case, whereas in the unweighted case such conditions seem to be unavoidable (see [31]). To see the point, note that Theorem 7.3 does not cover the simple

problem

$$\begin{cases} du(t, x) = \frac{1}{2}\Delta u(t, x) dt + dw(t), & t \in [0, T], x \in (0, 1), \\ u(t, 0) = u(t, 1) = 0, & t \in [0, T], \\ u(0, x) = 0, & x \in (0, 1), \end{cases}$$

in, say,  $E = L^q(0, 1)$  with  $q \in (2, \infty)$ , where  $w$  is a real-valued Brownian motion. Now the constant function  $g = \mathbf{1}$  in the noise term  $dw(t) = \mathbf{1} dw(t)$  does not belong to  $\mathcal{D}((-\Delta)^{\frac{1}{2}}) = H_0^{1,q}(0, 1)$  due to the Dirichlet boundary conditions. We refer to [50, Section 4] for a further discussion of this example.

It seems likely that, under suitable regularity assumptions on the coefficients, the  $L^p$ -realisations of the operators  $\mathcal{A}$  on weighted domains should have a bounded  $H^\infty$ -calculus. If true, the weighted domain case could be treated by our methods as well. Maximal  $L^p$ -regularity results for elliptic operators on  $L^p(\mathcal{O}, w)$  for open domains  $\mathcal{O} \subseteq \mathbb{R}^d$  and Muckenaupt weights  $w$  were proved in [36]. Further evidence is provided by the fact (see [23]) that if the linear Cauchy problem with additive noise has maximal regularity for both  $A$  and its adjoint, then  $A$  necessarily has a bounded  $H^\infty$ -calculus.

## 8. THE STOCHASTIC NAVIER-STOKES EQUATION

Let  $d \geq 2$  be a fixed integer and suppose that  $\mathcal{O}$  is a smooth bounded open domain in  $\mathbb{R}^d$ . Let  $H$  be a Hilbert space (for instance  $\ell^2$  or  $L^2(\mathcal{O})$ ). Let  $q \in (1, \infty)$  be fixed. We are interested in local existence of strong solutions in  $(H^{1,q}(\mathcal{O}))^d$  of the Navier-Stokes equation

$$(8.1) \quad \begin{cases} \frac{\partial u}{\partial t} = \Delta u - (u \cdot \nabla)u + f^0 - \nabla p + (g(u, \nabla u) + g^0)\dot{W}_H, \\ \operatorname{div} u(t, \cdot) = 0, & t > 0, \\ u(t, x) = 0, & t > 0, x \in \partial\mathcal{O}, \\ u(0, \cdot) = u_0. \end{cases}$$

Note that we allow  $g$  to depend on both  $u$  and  $\nabla u$ . As is well known (see, for instance, [8, 63]) such dependencies arise in the modelling of the onset of turbulence.

The function  $u_0 : \mathcal{O} \rightarrow \mathbb{R}^d$  is the initial velocity field,  $W_H$  is a cylindrical Brownian motion in  $H$ , and  $u$  and  $p$  represent the velocity field and the pressure of the fluid, respectively. We assume that  $f^0$  and  $g^0$  are strongly measurable and adapted and belong to  $L^1(0, T; H^{-1,q}(\mathcal{O}))$  and  $L^2(0, T; L^q(\mathcal{O}; H))$  almost surely, respectively. The function  $g$  is interpreted as a strongly measurable mapping

$$g : (H^{1,q}(\mathcal{O}))^d \rightarrow (L^q(\mathcal{O}; H))^d,$$

and we assume that for and all  $x, y \in (H^{1,q}(\mathcal{O}))^d$  we have

$$(8.2) \quad \|g(u) - g(v)\|_{(L^q(\mathcal{O}; H))^d} \leq L_g \|u - v\|_{(H^{1,q}(\mathcal{O}))^d} + \tilde{L}_g \|u - v\|_{(L^q(\mathcal{O}))^d}.$$

It is well known (see [32] and [40, Section 9]) that we have the direct sum decomposition

$$(L^q(\mathcal{O}))^d = \mathbb{X}^q \oplus \mathbb{G}^q,$$

where  $\mathbb{X}^q$  is the closure in  $(L^q(\mathcal{O}))^d$  of the set  $\{u \in (C_c^\infty(\mathcal{O}))^d : \nabla \cdot u = 0\}$  and  $\mathbb{G}^q = \{\nabla p : p \in H^{1,q}(\mathcal{O})\}$ . We denote by  $P$  the Helmholtz projection from

$(L^q(\mathcal{O}))^d$  onto  $\mathbb{X}^q$  along this decomposition. The negative *Stokes operator* is the linear operator  $(A, D(A))$  defined by

$$\begin{aligned} D(A) &= \mathbb{X}^q \cap D(\Delta_{\text{Dir}}), \\ Av &= -P(\Delta u), \quad u \in D(A), \end{aligned}$$

where  $D(\Delta_{\text{Dir}})$  is the domain of the Dirichlet Laplacian in  $(L^q(\mathcal{O}))^d$ , which for  $C^2$ -domains equals

$$D(\Delta_{\text{Dir}}) = \{u \in (H^{2,q}(\mathcal{O}))^d : u = 0 \text{ on } \partial\mathcal{O}\}.$$

The operator  $A$  is boundedly invertible (see [15] and [40, page 797]),  $-A$  generates a bounded analytic  $C_0$ -semigroup in  $\mathbb{X}^q$ , and it was shown in [71] (for  $C^3$  domains) and [40, Theorem 9.17] (for  $C^{1,1}$  domains) that  $A$  has a bounded  $H^\infty$ -calculus on  $\mathbb{X}^q$ :

**Proposition 8.1.** *For all  $q \in (1, \infty)$  the negative Stokes operator  $A$  has a bounded  $H^\infty(\Sigma_\sigma)$ -calculus of angle  $0 < \sigma < \frac{1}{2}\pi$  on  $\mathbb{X}^q$ .*

It is well known (see [81] for the details) that, by applying the Helmholtz projection  $P$  to  $u$ , the Navier-Stokes equation (8.1) can be reformulated as an abstract stochastic evolution on

$$X_0 := \mathbb{X}_{-\frac{1}{2}}^q,$$

where the space on the right-hand side is defined as the completion of  $\mathbb{X}^q$  with respect to the norm

$$\|x\|_{X_0} := \|A^{-\frac{1}{2}}x\|_{\mathbb{X}^q}.$$

In particular, as a Banach space,  $X_0$  is isomorphic to a closed subspace of  $L^q(\mathcal{O})$ .

The bounded invertibility of  $A$  implies that the identity operator on  $\mathbb{X}^q$  extends to a continuous embedding  $\mathbb{X}^q \hookrightarrow X_0$ . Furthermore we set

$$X_1 := D(A^{\frac{1}{2}}).$$

For  $s \in (0, 1]$  and  $s - \frac{1}{q} > 0$  let  $H_0^{s,q}(\mathcal{O})$  and  $B_{q,p,0}^s(\mathcal{O})$  denote the closed subspaces of  $H^{s,q}(\mathcal{O})$  and  $B_{q,p}^s(\mathcal{O})$  with zero trace. If  $s - \frac{1}{q} < 0$  we let  $H_0^{s,q}(\mathcal{O}) = H^{s,q}(\mathcal{O})$  and  $B_{q,p,0}^s(\mathcal{O}) = B_{q,p}^s(\mathcal{O})$ . Furthermore let  $H^{-1,q}(\mathcal{O})$  be the dual of  $H_0^{1,q'}(\mathcal{O})$  with  $\frac{1}{q} + \frac{1}{q'} = 1$ .

The following lemma is well known.

**Lemma 8.2.** *For every  $\alpha \in [\frac{1}{2}, 1]$  and  $p, q \in (1, \infty)$  with  $2\alpha - 1 - \frac{1}{q} \neq 0$  one has*

$$(8.3) \quad X_\alpha = D(A^{\alpha-\frac{1}{2}}) = \mathbb{X}^q \cap (H_0^{2\alpha-1,q}(\mathcal{O}))^d,$$

$$(8.4) \quad X_{\alpha,p} = \mathbb{X}^q \cap (B_{q,p,0}^{2\alpha-1}(\mathcal{O}))^d.$$

Moreover,  $P$  induces a bounded linear operator

$$P : (H^{-1,q}(\mathcal{O}))^d \rightarrow X_0.$$

*Proof.* To prove (8.3) note that

$$(8.5) \quad X_\alpha = D(A^{\alpha-\frac{1}{2}}) = [\mathbb{X}^q, D(A)]_{\alpha-\frac{1}{2}},$$

where we used [2, Theorem V.1.5.4], Proposition 8.1 and (3.2). The second identity in (8.3) follows from [33], [40, Theorem 9.17] and [84, Theorem 1.17.1.1]. By a similar reasoning one obtains (8.4).

The final assertion follows from a similar argument as in [40, Proposition 9.14] (see also [52, Proposition 3.1]).  $\square$

Define  $F : X_{\theta+\frac{1}{2}} \times X_{\theta+\frac{1}{2}} \rightarrow X_0$  by

$$F(u, v) = -P((u \cdot \nabla)v)$$

and write  $F(u) := F(u, u)$ . We will check that these mappings are well defined for  $\theta \geq \frac{d}{4q}$ . Indeed, by [34], for these  $\theta$  one has

$$\|A^{-\frac{1}{2}}F(u, v)\|_{L^q(\mathcal{O})} \leq C\|A^\theta u\|_{L^q(\mathcal{O})}\|A^\theta v\|_{L^q(\mathcal{O})}, \quad u, v \in \mathbf{D}(A^\theta).$$

This can be reformulated as

$$\|F(u, v)\|_{X_0} \leq C\|u\|_{X_{\theta+\frac{1}{2}}}\|v\|_{X_{\theta+\frac{1}{2}}}, \quad u, v \in X_{\theta+\frac{1}{2}},$$

from which the well-definedness follows. Moreover, one immediately obtains the following local Lipschitz estimate (see [12])

$$\|A^{-\frac{1}{2}}(F(u) - F(v))\|_{L^q(\mathcal{O})} \leq C(\|A^\theta u\|_{L^q(\mathcal{O})} + \|A^\theta v\|_{L^q(\mathcal{O})})\|A^\theta u - A^\theta v\|_{L^q(\mathcal{O})},$$

which can be reformulated as

$$\|F(u) - F(v)\|_{X_0} \leq C(\|u\|_{X_{\theta+\frac{1}{2}}} + \|v\|_{X_{\theta+\frac{1}{2}}})\|u - v\|_{X_{\theta+\frac{1}{2}}}.$$

In particular, if  $0 \leq \theta < \frac{1}{2} - \frac{1}{p}$  and  $p \in [2, \infty)$ , then  $1 - \frac{1}{p} > \theta + \frac{1}{2}$  and therefore  $F : X_{1-\frac{1}{p}, p} \times X_{1-\frac{1}{p}, p} \rightarrow X_0$  is locally Lipschitz continuous.

Next define  $B : X_1 \rightarrow \gamma(H, X_{\frac{1}{2}})$  by

$$B(u) = P(g(u)).$$

This is well defined, because  $g$  maps  $X_1 = (H^{1,q}(\mathcal{O}))^d$  into

$$(L^q(\mathcal{O}; H))^d = (\gamma(H, L^q(\mathcal{O}))^d = \gamma(H, (L^q(\mathcal{O}))^d),$$

and the Helmholtz projection extends to a bounded projection  $P : \gamma(H, (L^q(\mathcal{O}))^d) \rightarrow \gamma(H, \mathbb{X}^q) = \gamma(H, X_{\frac{1}{2}})$  in a canonical way. Here we used (2.1) and (8.5).

Now we can reformulate (8.1) as an abstract stochastic evolution equation in  $X_0$  of the form

$$(8.6) \quad \begin{cases} dU(t) + \mathcal{A}U(t) dt = [F(U(t)) + f(t)] dt + [B(U(t)) + b(t)] dW_H(t), \\ U(0) = u_0, \end{cases}$$

where  $\mathcal{A} = A_{-\frac{1}{2}}$ ,  $f = Pf^0$  and  $b = Pg^0$ .

**Theorem 8.3.** *Let  $d \geq 2$ , and let  $p > 2$  and  $q \geq 2$  satisfy  $\frac{d}{2q} < 1 - \frac{2}{p}$ . Let  $u_0 : \Omega \rightarrow \mathbb{X}^q \cap B_{q,p,0}^{1-\frac{2}{p}}(\mathcal{O})^d$  be strongly  $\mathcal{F}_0$ -measurable. Let  $f^0 \in L^0_{\mathcal{F}}(\Omega; L^p(0, T; (H^{-1,q}(\mathcal{O}))^d))$  and  $g^0 \in L^0_{\mathcal{F}}(\Omega; L^p(0, T; L^q(\mathcal{O}; H)))$ . If the Lipschitz constant  $L_g$  in (8.2) is small enough, then the problem (8.6) admits a unique maximal local mild solution on  $[0, T]$  with values in  $(H_0^{1,q}(\mathcal{O}))^d$ . Moreover, this solution has a modification with continuous trajectories in  $(B_{q,p,0}^{1-\frac{2}{p}}(\mathcal{O}))^d$ .*

*Proof.* The operator family  $\mathcal{J}$  is  $R$ -bounded. Furthermore, by Proposition 8.1  $A$  has a bounded  $H^\infty$ -calculus on  $\mathbb{X}^q = X_{\frac{1}{2}}$  (the equality of these spaces follows from (8.3)) of angle  $< \frac{1}{2}\pi$ . Therefore,  $\mathcal{A} = A_{-\frac{1}{2}}$  has a bounded  $H^\infty$ -calculus on  $X_0$ .

By Lemma 8.2 (and noting that  $1 - \frac{2}{p} > \frac{d}{2q} \geq \frac{1}{q}$  to justify the boundary conditions), one has  $X_0 = \mathbb{X}^q \cap (H^{-1,q}(\mathcal{O}))^d$   $X_1 = \mathbb{X}^q \cap (H_0^{1,q}(\mathcal{O}))^d$  and

$$(X_0, X_1)_{1-\frac{1}{p}, p} = \mathbb{X}^q \cap (B_{q,p,0}^{1-\frac{2}{p}}(\mathcal{O}))^d, \quad (X_0, X_1)_{\frac{1}{2}} = \mathbb{X}^q.$$

By Lemma 8.2,  $u_0 \in (X_0, X_1)_{1-\frac{1}{p}, p}$  almost surely.

For any  $\theta \in [\frac{d}{4q}, \frac{1}{2} - \frac{1}{p})$ , we can apply Theorem 5.6 with  $F^{(1)} = 0$ ,  $F^{(2)} = F$ ,  $B^{(1)} = B$ , and  $B^{(2)} = 0$  (and combine (8.2) with Remark 4.1 to check the assumptions concerning  $B^{(1)}$ ) to obtain a unique maximal local mild solution  $U$  which satisfies the assertions of Theorem 5.6.  $\square$

*Remark 8.4.* The above result is merely a proof-of-principle and can be extended into various directions. For instance, more general ranges of the parameters can be considered as in [12, 34]; different regularity assumptions on the coefficients are possible, and different regularity of the solutions will result. Furthermore, we expect global existence in dimension  $d = 2$ . Using the results of [51, 52], we believe that it should be possible to adapt the above techniques to study maximal regularity for the Navier–Stokes equation on  $\mathbb{R}^d$  (see also the discussion below). Along similar lines, it should be possible to use the results of [51, 52, 71] to study maximal regularity in the case of exterior domains in  $\mathbb{R}^d$ . We plan to address such issues in a forthcoming paper.

**8.1. Discussion.** The existence of  $H^{1,q}(\mathcal{O})$ -solutions for the stochastic Navier–Stokes equation in dimension  $d = 2$  was established, under a trace class assumption on the noise replacing our assumption on  $g$ , by Brzeźniak and Peszat [12]. In their framework,  $g$  is a  $C^1$ -function on  $\mathbb{R}^d$  with locally Lipschitz continuous derivatives; it is then shown that  $g$  induces a locally Lipschitz continuous mapping  $G$  from  $X_\eta$  to  $\gamma(H, X_{\frac{1}{2}})$  for suitable exponents  $\eta > 1$ . However,  $G$  is not defined on  $X_1$  and therefore  $g$  cannot be allowed to depend on both  $u$  and  $\nabla u$ .

Under the same assumptions on  $g$  as ours, existence of a local strong  $H^{1,q}(\mathbb{R}^d)$ -solution for dimensions  $d \geq 2$  has been shown by Mikulevicius and Rozovskii [63]. Existence and uniqueness of local strong  $H^{1,2}$ -solutions in bounded domains was obtained by Mikulevicius [61]. In both papers, global existence for  $d = 2$  is established as well.

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