ON SEMI-R-BOUNDEDNESS AND ITS APPLICATIONS

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Abstract. R-Boundedness is a randomized boundedness condition for sets of operators which in recent years has found many applications in the maximal regularity theory of evolution equations, stochastic evolution equations, spectral theory and vector-valued harmonic analysis. However, in some situations additional geometric properties such as Pisier's property (α) are required to guaranty the R-boundedness of a relevant set of operators. In this paper we show that a weaker property called semi-R-boundedness can be used to avoid these geometric assumptions in the context of Schauder decompositions and the H^∞-calculus. Furthermore, we give weaker conditions for stochastic integrability of certain convolutions.

1. Introduction

R-boundedness has proved to be an important tool in the theory of maximal regularity of evolution equations [38], in operator theory [21], Schauder decompositions [3, 6], vector-valued harmonic analysis [12, 17] and stochastic equations (see [28] and references therein). In particular from the above results one can see that many results for Hilbert spaces extend to the Banach space setting if one replaces uniform boundedness by R-boundedness. There are situations in which additional geometric assumptions such as Pisier's property (α) are required to guaranty the R-boundedness of certain sets of operators (see [18, 25]). We show that in several situations these assumptions can be avoided by using the weaker notion of semi-R-boundedness.

Definition 1.1. *Let X and Y be Banach spaces. Let $(r_n)_{n\geq 1}$ be a Rademacher sequence on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A collection $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ is said to be* R-bounded *if there exists a constant $M \geq 0$ such that*

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n T_n x_n\Big\|^2\Big)^{\frac12} \leq M\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n x_n\Big\|^2\Big)^{\frac12}, \tag{1.1}$$

for all $N \geq 1$ and all sequences $(T_n)_{n=1}^N$ in $\mathcal{T}$ and $(x_n)_{n=1}^N$ in X.

By the Kahane-Khintchine inequalities one can replace the $L^2(\Omega; E)$-norm in (1.1) by any $L^p(\Omega; E)$-norm as long as $p \in [1, \infty)$.

If one only considers x_n of the form $x_n = a_n x$, where a_n is a scalar and $x \in X$, then one obtains the weaker notion of semi-R-boundedness:

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Definition 1.2. *A collection $\mathcal{T} \subseteq \mathcal{L}(X,Y)$ is said to be* semi-R-bounded *if there exists a constant $M \geq 0$ such that*

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n T_n a_n x\Big\|^2\Big)^{\frac12} \leq M\Big(\sum_{n=1}^N |a_n|^2\Big)^{\frac12}\|x\|, \tag{1.2}$$

for all $N \geq 1$ and all sequences $(T_n)_{n=1}^N$ in $\mathcal{T}$, scalars $(a_n)_{n=1}^N$, and $x \in X$.

This notion has been introduced and studied in [14]. In this paper we provide several characterizations and applications of semi-R-boundedness. Let us note that semi-R-boundedness is used in [4] to compare different operator norms. The maximal function which was used to study Kato's square root in an L^p-setting in [15] is also defined in terms of semi-R-boundedness.

In this paper we give further properties and characterizations of semi-R-boundedness (see Sections 2 and 4) which prepare us for our main applications.

In Section 3 we give sufficient conditions for semi-R-boundedness in terms of smoothness of operators. We provide semi-R-bounded versions of results in [16] and prove sharp results for semigroups. Applications to stochastic equations are given in Section 5. Here we apply multiplier and factorization techniques to obtain path-continuity of solutions. In Section 6 we prove that the partial sum projections in a Schauder decomposition are always semi-R-bounded. Under geometric constrictions on the Banach space R-boundedness results were obtained in [32]. Finally, in Section 7 we characterize the boundedness of the H^∞-calculus in terms of semi-R-bounded imaginary powers. Such results were known in the Hilbert space situation and for Banach spaces with so-called property (α) (see [22, 23, 26, 39]). We obtain a characterization for spaces with nontrivial type and also show that the H^∞-calculus itself is semi-R-bounded.

We will write $a \lesssim b$ if there exists a universal constant $C > 0$ such that $a \leq Cb$, and $a \eqsim b$ if $a \lesssim b \lesssim a$. If we want to emphasize that C depends on some parameter t, we write $a \lesssim_t b$ and $a \eqsim_t b$.

2. Definitions and basic properties

Let X and Y be Banach spaces. Let $(r_n)_{n\geq1}$ be a Rademacher sequence and $(\gamma_n)_{n\geq1}$ be a Gaussian sequence on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $\mathcal{T} \subseteq \mathcal{L}(X,Y)$. The least constant M for which (1.1) in Definition 1.1 holds is called the *R-bound* of $\mathcal{T}$, and is denoted by $R(\mathcal{T})$. Replacing the Rademacher sequence $(r_n)_{n\geq1}$ by a Gaussian sequence $(\gamma_n)_{n\geq1}$ one obtains the definition of γ-boundedness. The least corresponding constant is denoted by $\gamma(\mathcal{T})$ and is called the *γ-bound* of $\mathcal{T}$. For more details on R-boundedness we refer to [6, 38]. If a family $\mathcal{T} \subset \mathcal{L}(X,Y)$ is R-bounded then it is γ-bounded and $\gamma(\mathcal{T}) \leq R(\mathcal{T})$. If X has finite cotype then γ-boundedness and R-boundedness of $\mathcal{T}$ are equivalent and $R(\mathcal{T}) \leq C_X\gamma(\mathcal{T})$. For details on type and cotype we refer to [9].

The least constant M for which (1.2) in Definition 1.2 holds is called the *semi-R-bound* of $\mathcal{T}$ and is denoted by $R_{\rm s}(\mathcal{T})$. If we replace $(r_n)_{n\geq1}$ by $(\gamma_n)_{n\geq1}$, the least corresponding constant M is denoted by $\gamma_{\rm s}(\mathcal{T})$. Both conditions imply uniform boundedness with $\sup_{T\in\mathcal{T}}\|T\| \leq \gamma_{\rm s}(\mathcal{T}) \leq R_{\rm s}(\mathcal{T})$. Clearly, R-boundedness implies semi-R-boundedness and γ-boundedness implies semi-γ-boundedness. Moreover, by a standard randomization argument and [9, Proposition 12.11] one can see that semi-R-boundedness and semi-γ-boundedness are equivalent and

$$\gamma_{\mathrm{s}}(\mathcal{T}) \leq R_{\mathrm{s}}(\mathcal{T}) \leq \sqrt{\pi/2}\,\gamma_{\mathrm{s}}(\mathcal{T}). \tag{2.1}$$

Note that $X = \mathcal{L}(\mathbb{K}, X)$, where we associate to each $x \in X$ the operator $a \mapsto ax$. The following trivial but useful observation will allow us to reduce questions on semi-R-boundedness to the well-known situation of R-boundedness.

Lemma 2.1. *Let X and Y be Banach spaces. For a collection $\mathcal{T} \subseteq \mathcal{L}(X,Y)$ the following assertions are equivalent:*

1. *$\mathcal{T}$ is semi-R-bounded with $R_{\mathrm{s}}(\mathcal{T}) \leq M$.*
2. *For all $x \in X$, the set $\mathcal{T}_x = \{Tx \in \mathcal{L}(\mathbb{K}; Y) : T \in \mathcal{T}\}$ is R-bounded with $R(\mathcal{T}_x) \leq M\|x\|$.*

The following result can be proved as in [14, Proposition 2.1] where the case $X = Y$ was considered.

Proposition 2.2. *Let X be a non-zero Banach space and let Y be a Banach space. The following assertions are equivalent:*

1. *Y is of type 2.*
2. *Every uniformly bounded collection $\mathcal{T} \subset \mathcal{L}(X,Y)$ is semi-R-bounded.*

In particular, this result implies that there are many collections $\mathcal{T} \subset \mathcal{L}(X,Y)$ which are semi-R-bounded but not R-bounded.

Example 2.3. Let $p \in [2,\infty)$ and let $(S(t))_{t\geq 0}$ be the translation group on $L^p(\mathbb{R})$. Then $\{S(t) : t \in [-1,1]\}$ is semi-R-bounded but not R-bounded (cf. [16, Example 6.2] and see Example 3.5 below for related results).

The following results can all be obtained from Lemma 2.1 and the corresponding R-boundedness result which can be found in [6, 38]. We only state the results that we need.

Remark 2.4. Let $\mathcal{T} \subset \mathcal{L}(X,Y)$ be a collection. The following hold:

1. If $\mathcal{T}$ is semi-R-bounded, then the absolute convex hull abs co$(\mathcal{T})$ of $\mathcal{T}$ is semi-R-bounded and $R_{\mathrm{s}}(\text{abs co}(\mathcal{T})) \leq 2R_{\mathrm{s}}(\mathcal{T})$
2. If $\mathcal{T}$ is semi-R-bounded, then the strong closure of $\mathcal{T}$ is semi-R-bounded and $R_{\mathrm{s}}(\overline{\mathcal{T}}^{\text{strong}}) \leq R_{\mathrm{s}}(\mathcal{T})$.
3. In the definition of semi-R-boundedness it suffices to take the operators $T_1, \ldots, T_N$ distinct.

A space X is said to be a *Grothendieck space* (GT space) if every $T : X \to \ell^2$ is 1-summing (cf. [9] for details on summing operators). Recall that L^1-spaces are GT spaces.

Proposition 2.5. *Let X be a Banach space, and let Y be a non-zero Banach space.*

1. *If X has cotype 2 and is a GT space, then every semi-R-bounded family $\mathcal{T}$ is R-bounded.*
2. *If every semi-R-bounded collection $\mathcal{T} \subset \mathcal{L}(X,Y)$ is R-bounded, then X has cotype 2.*

The result in the case $X = Y$ is separable has been proved in [14, Theorem 2.2] and the more general case has been considered in [4, Proposition 3.17]. In the case $X = Y$ is a separable Banach space, a complete characterization of spaces for which semi-R-boundedness and R-boundedness coincide has been given in [14, Theorem 2.2].

One can weaken the definition of semi-R-boundedness by taking $a_n = 1$ for all n. The next result shows that this is in fact equivalent to semi-R-boundedness.

Proposition 2.6. *Let X and Y be Banach spaces. For a collection $\mathcal{T} \subseteq \mathcal{L}(X,Y)$ the following assertions are equivalent:*

(1) *The collection $\mathcal{T}$ is semi-R-bounded.*
(2) *The collection $\mathcal{T}$ is semi-γ-bounded.*
(3) *There exists an $M \geq 0$ such that $\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n T_n x\Big\|^2\Big)^{\frac12} \leq M\sqrt{N}\|x\|$ for all $N \geq 1$, $(T_n)_{n=1}^N$ in $\mathcal{T}$ and $x \in X$.*
(4) *There exists an $M \geq 0$ such that $\Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n T_n x\Big\|^2\Big)^{\frac12} \leq M\sqrt{N}\|x\|$, for all $N \geq 1$, $(T_n)_{n=1}^N$ in $\mathcal{T}$ and $x \in X$.*

Moreover, $\gamma_{\rm s}(\mathcal{T}) \leq R_{\rm s}(\mathcal{T}) \leq \sqrt{\frac{\pi}{2}}\gamma_{\rm s}(\mathcal{T})$, and $2^{-\frac12}\gamma_{\rm s}(\mathcal{T}) \leq M_\gamma \leq M_r \leq R_{\rm s}(\mathcal{T})$, where M_r and M_γ are the least constants for which statements (3) and (4) above hold.

Proof. For (1)⇔(2) see (2.1). (2)⇒(4) and (1)⇒(3) are trivial.

(4)⇒(2): The proof is based on an approximation argument (see [19]). We first consider the case $(a_n)_{n=1}^N$ in $\mathbb{R}$. By symmetry we may assume $a_n \geq 0$ for all n. By an approximation argument it is enough to consider positive $(a_n)_{n=1}^N$ in $\mathbb{Q}$. We can find integers $K \geq 1$ and $(p_n)_{n=1}^N$ in $\mathbb{N}$ such that $a_n = \frac{p_n}{K}$ for all n. Let $(\gamma_{nm})_{n,m\geq 1}$ be a Gaussian sequence. Since $(p_n\gamma_n)_{n=1}^N$ and $(\sum_{m=1}^{p_n^2}\gamma_{nm})_{n=1}^N$ are identically distributed, we have

$$\begin{aligned}\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n T_n a_n x\Big\|^2 &= \frac{1}{K^2}\mathbb{E}\Big\|\sum_{n=1}^N p_n\gamma_n T_n x\Big\|^2 = \frac{1}{K^2}\mathbb{E}\Big\|\sum_{n=1}^N\sum_{m=1}^{p_n^2}\gamma_{nm}T_n x\Big\|^2\\ &\leq \frac{M^2}{K^2}\mathbb{E}\Big\|\sum_{n=1}^N\sum_{m=1}^{p_n^2}\gamma_{nm}x\Big\|^2 = \frac{M^2}{K^2}\mathbb{E}\Big\|\sum_{n=1}^N p_n\gamma_n x\Big\|^2 = M^2\mathbb{E}\Big\|\sum_{n=1}^N\gamma_n a_n x\Big\|^2.\end{aligned}$$

For $(a_n)_{n=1}^N$ in $\mathbb{C}$ we can consider the real and imaginary part separately to obtain

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n T_n a_n x\Big\|^2\Big)^{\frac12} \leq \sqrt{2}M\|x\|\Big(\sum_{n=1}^N |a_n|^2\Big)^{1/2}.$$

(3)⇒(4): The result follows from a standard central limit theorem argument. Indeed, let $(r_{nk})_{n,k\geq 1}$ be a Rademacher sequence. Let $N \geq 1$ and $(x_n)_{n=1}^N$ in X be arbitrary. One has

$$\begin{aligned}\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n T_n x\Big\|^2 &= \lim_{K\to\infty}\frac{1}{K}\mathbb{E}\Big\|\sum_{k=1}^K\sum_{n=1}^N r_{nk}T_n x\Big\|^2\\ &\leq M^2\lim_{K\to\infty}\frac{1}{K}\mathbb{E}\Big\|\sum_{k=1}^K\sum_{n=1}^N r_{nk}x\Big\|^2 = M^2\sqrt{N}\|x\|.\end{aligned}$$

□

3. Smooth operator-valued functions

In this section we show that under type and cotype assumptions certain smooth operator-valued functions have semi-R-bounded range. The case of R-boundedness has been considered in [16, Theorem 5.1]. The smoothness below is expressed in Besov and Hölder spaces. Details on Besov spaces and other spaces can be found in [36] (see [1, 35] for the vector-valued setting). For details on type and cotype we refer to [9] and references therein.

Theorem 3.1. *Let X and Y be Banach spaces. Let $p \in [1,2]$. Assume that Y has type p. Let $T : \mathbb{R}^d \to \mathcal{L}(X,Y)$ be strongly continuous. Let $r \in [2,\infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$ and assume that there is an M such that for all $x \in X$,*

$$\|Tx\|_{B^{\frac{d}{r}}_{r,1}(\mathbb{R}^d;Y)} \leq M\|x\|.$$

Then there exists a constant $C = C(p,Y)$ such that

$$R_{\mathrm{s}}\big(\{T(t) \in \mathcal{L}(X,Y) : t \in \mathbb{R}^d\}\big) \leq CM. \tag{3.1}$$

Remark 3.2.

(1) Note that $B^{\frac{d}{r}}_{r,1}(\mathbb{R}^d;Y) \hookrightarrow BUC(\mathbb{R}^d;Y)$, so that $\{T(t) : t \in \mathbb{R}^d\}$ is always uniformly bounded.

(2) Theorem 3.1 also holds if T is defined on a smooth domain $D \subset \mathbb{R}^d$. This easily follows from the boundedness of the extension operator (cf. [36]).

Proof of Theorem 3.1. Let $x \in X$ be arbitrary. Then $\{Tx \in B^{\frac{d}{r}}_{r,1}(\mathbb{R}^d;\mathcal{L}(\mathbb{K},Y))\}$ and [16, Theorem 5.1] implies that $\{T(t)x \in \mathcal{L}(\mathbb{K},Y) : t \in \mathbb{R}^d\}$ is R-bounded by $C(p,Y)\|x\|$. Therefore, Lemma 2.1 gives that $\{T(t) \in \mathcal{L}(X,Y) : t \in \mathbb{R}^d\}$ is semi-R-bounded by $C(p,Y)$. $\square$

As a consequence we obtain that Hölder regularity of an operator-valued function implies semi-R-boundedness which in our situation can be proved in the same way as [11, Corollary 5.4]

Corollary 3.3. *Let X and Y be Banach spaces. Let $p \in [1,2)$. Assume that Y has type p. Let $I = (a,b)$ with $-\infty \leq a < b \leq \infty$. Let $\alpha > 0$ be such that $\alpha > \frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. Assume $T : \mathbb{R} \to \mathcal{L}(X,Y)$ and M are such that for all $x \in X$, $\|Tx\|_{L^r(I;Y)} \leq M\|x\|$ and there exists an A such that*

$$\|T(s+h)x - T(s)x\| \leq A|h|^\alpha(1+|s|)^{-\alpha}\|x\|, \quad s, s+h \in I,\ h \in I, x \in X. \tag{3.2}$$

Then $\{T(t) \in \mathcal{L}(X,Y) : t \in I\}$ is semi-R-bounded by a constant times A.

Note that in the case where I is bounded, the factor $(1+|s|)^{-\alpha}$ can be omitted. The situation $p = 2$ omitted as it is covered by Proposition 2.2.

Next, we prove a result on semi-R-boundedness of strongly continuous semigroups restricted to real interpolation spaces. The result is sharp in the smoothness index. A similar result for R-boundedness has been obtained in [16, Theorem 6.1]. However, there it is unclear what happens for the sharp exponent in the smoothness index.

For details on semigroups and interpolation theory we refer to [10] and [2, 36].

Corollary 3.4. *Let $(S(t))_{t\in\mathbb{R}_+}$ be a strongly continuous semigroup on a Banach space X with $\|S(t)\| \leq Me^{-\omega t}$ for some $M, \omega > 0$. Assume X has type $p \in [1,2)$. Let $\alpha = \frac{1}{p} - \frac{1}{2}$ and let $i_\alpha : (X, D(A))_{\alpha,1} \to X$ be the inclusion mapping. Then*

$$\{S(t)i_\alpha : t \in \mathbb{R}_+\} \subset \mathcal{L}((X, D(A))_{\alpha,1}, X)$$

is semi-R-bounded.

If S is not exponentially stable, then one obtains that for all $K > 0$ the set

$$\{S(t)i_\alpha : t \in [0, K]\} \subset \mathcal{L}((X, D(A))_{\alpha,1}, X)$$

is semi-R-bounded. This follows from Corollary 3.4 and a translation argument.

Proof. Let $p \in [1,2)$. Recall from [2, Theorem 6.7.3] that $x \in (X, D(A))_{\alpha,1}$ if and only if $x \in X$ and

$$\|x\|_{\alpha,1} = \|x\| + \int_0^\infty t^{-\alpha} \sup_{0\leq h\leq t} \|S(h)x - x\| \frac{dt}{t} < \infty.$$

Moreover, $\|\cdot\|_{\alpha,1}$ defines an equivalent norm on $(X, D(A))_{\alpha,1}$.

Let $N : \mathbb{R} \to \mathcal{L}(X, D(A))_{\alpha,1}, X)$ be given by $N(t) = S(|t|)i_\alpha$. Let $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. By [33, Proposition 3.1]

$$\|Nx\|_{B^\alpha_{r,1}(\mathbb{R};X)} \eqsim_r \|Nx\|_{L^r(\mathbb{R};X)} + \int_0^\infty t^{-\alpha} \sup_{|h|\leq t} \|N(\cdot + h)x - N(\cdot)x\|_{L^r(\mathbb{R};X)} \frac{dt}{t}.$$

Since S is exponentially stable, $\|Nx\|_{L^r(\mathbb{R};X)} \lesssim_r \|x\|$. For the other term, using the semigroup property we get

$$\|N(\cdot + h)x - N(\cdot)x\|^r_{L^r(\mathbb{R};X)} \lesssim_{M,\omega,r} \sup_{|h|\leq t} \|(N(h)x - x)\|^r.$$

Since $N(h) = N(-h)$ it follows that

$$\int_0^\infty t^{-\alpha} \sup_{|h|\leq t} \|N(\cdot + h)x - N(\cdot)x\|_{L^r(\mathbb{R};X)} \frac{dt}{t} \lesssim_{M,\omega,r} \int_0^\infty t^{-\alpha} \sup_{0\leq h\leq t} \|S(h)x - x\| \frac{dt}{t}.$$

Therefore, $\|Nx\|_{B^\alpha_{r,1}(\mathbb{R};X)} \lesssim_{M,\omega,r,\alpha} \|x\|_{\alpha,1}$ and the result follows from Theorem 3.1. $\square$

In the next example we show that the result in Corollary 3.4 is sharp.

Example 3.5. Let $p \in [1,2)$. Let $(S(t))_{t\in\mathbb{R}}$ be the left-translation group on $X = L^p(\mathbb{R})$ with generator $A = \frac{d}{dx}$.

(1) Let $\alpha = \frac{1}{p} - \frac{1}{2}$. Then for all $K \in \mathbb{R}_+$,

$$\{S(t)i_\alpha : t \in [-K, K]\} \subset \mathcal{L}(B^\alpha_{p,1}(\mathbb{R}), L^p(\mathbb{R})), \tag{3.3}$$

is semi-R-bounded. Here $i_\alpha : B^\alpha_{p,1}(\mathbb{R}) \to L^p(\mathbb{R})$ denotes the canonical embedding. This result follows from Corollary 3.4 and $(X, D(A))_{\alpha,1} = B^\alpha_{p,1}(\mathbb{R})$ (cf. [35, Theorems 4.2 and 4.3.3])

(2) For $\alpha \in [0, \frac{1}{p} - \frac{1}{2})$ and $K = 1$, the family (3.3) is not semi-R-bounded. This follows from the proof of [16, Example 6.2]

Remark 3.6. Note that if $\alpha > \frac{1}{p} - \frac{1}{2}$, then the family in (3.3) is even R-bounded (see [16, Example 6.2]). In general we do not know whether this extends to $\alpha = \frac{1}{p} - \frac{1}{2}$, but if $p = 1$ this is indeed the case. This follows from Proposition 2.5, since $B^{\frac{1}{2}}_{1,1}(\mathbb{R})$ (being an L^1 space) is a GT-space with cotype 2.

4. Multipliers in Gauss spaces

The next proposition is a semi-R-bounded version of the multiplier theorem [22, Proposition 4.11]. We present the result for the measure space $((a,b),\mu,\mathcal{B}_{(a,b)})$, where μ is the Lebesgue measure. The result is valid for more general measure spaces with the same proof, but we will need it only for intervals (a,b), where $-\infty\le a<b\le\infty$.

Let X be a Banach space and H be a separable Hilbert space with orthonormal basis $(h_n)_{n\ge1}$. Let $(\gamma_n)_{n\ge1}$ be a real-valued sequence of independent standard Gaussian random variables. An operator $R\in\mathcal{L}(H,X)$ is called γ-radonifying if $\sum_{n\ge1}\gamma_n Rh_n$ converges to some $\xi\in L^2(\Omega;X)$. Moreover, we let $\|R\|_{\gamma(H,X)}=\|\xi\|_{L^2(\Omega;X)}$.

Let $\phi:(a,b)\to\mathcal{L}(H,X)$ be strongly measurable and such that for all $x^*\in X^*$, $\phi^*x^*\in L^2(a,b;H)$. Let $I_\phi: L^2(a,b;H)\to X$ be the (Pettis)-integral operator given by

$$I_\phi f=\int_a^b \phi(t)f(t)\,dt.$$

We say $\phi\in\gamma(a,b;H,X)$ if $I_\phi: L^2(a,b;H)\to X$ is in $\gamma(L^2(a,b;H),X)$ (i.e. if I_ϕ is γ-radonifying). Note that we let $\gamma(a,b;X)=\gamma(a,b;\mathbb{R};X)$. For details on the Gauss spaces $\gamma(a,b;X)$ and $\gamma(a,b;H,X)$ we refer to [9, 22, 29].

Proposition 4.1. *Let X and Y be Banach spaces. Let $S:(a,b)\to\mathcal{L}(X,Y)$ be a strongly continuous map and let $\mathcal{S}=\{S(t)\in\mathcal{L}(X,Y):t\in(a,b)\}$. For a constant $K\ge0$, the following assertions are equivalent:*

(1) *$\mathcal{S}$ is semi-R-bounded with $\gamma_{semi}(\mathcal{S})\le K$,*
(2) *for all $x\in X$ and all $f\in L^2(a,b)$*

$$\|fSx\|_{\gamma(a,b;Y)}\le K\|f\|_{L^2(a,b)}\|x\|_X. \tag{4.1}$$

It actually suffices to consider indicator functions f in (4.1).

Proof. This follows from the Gaussian version of Lemma 2.1, [22, Proposition 4.11] and the fact that $\gamma(a,b;\mathbb{K})=L^2(a,b;\mathbb{K})$.

□

Proposition 4.2. *Let X and Y be Banach spaces and let H be a separable Hilbert space. Let $\mathcal{S}\subset\mathcal{L}(X,Y)$ be semi-R-bounded by some constant K. Then the set $\tilde{\mathcal{S}}\subset\mathcal{L}(\gamma(H,X),\gamma(H,Y))$ defined by*

$$\tilde{\mathcal{S}}=\{\tilde S:\exists S\in\mathcal{S}\text{ such that }\forall B\in\gamma(H,X)\text{ one has }\tilde S(B)=SB\},$$

is semi-R-bounded by K.

Proof. Let $S_1,\dots,S_N\in\mathcal{S}$ and $a_1,\dots,a_N\in\mathbb{K}$ be arbitrary. Then

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_nS_na_nB\Big\|^2_{\gamma(H,Y)}\Big)^{\frac12}&=\Big(\mathbb{E}\mathbb{E}_\gamma\Big\|\sum_{m\ge1}\gamma_m\sum_{n=1}^N r_nS_na_nBh_m\Big\|_Y^2\Big)^{\frac12}\\&=\Big(\mathbb{E}_\gamma\mathbb{E}\Big\|\sum_{n=1}^N r_nS_na_n\sum_{m\ge1}\gamma_mBh_m\Big\|_Y^2\Big)^{\frac12}\\&\le K\Big(\sum_{n=1}^N|a_n|^2\Big)^{\frac12}\|B\|_{\gamma(H,X)}\end{aligned}$$

and the result follows from Lemma 2.1. □

Let X be a Banach space. Let $(r_n)_{n\geq 1}$ and $(r'_n)_{n\geq 1}$ denote two independent Rademacher sequences and let $(r_{mn})_{m,n\geq 1}$ be a double indexed Rademacher sequence. We say that X has property (α^+) if there is a constant C such that for all $(x_{mn})_{m,n=1}^{M,N}$

$$\Big\|\sum_{n=1}^N\sum_{m=1}^M r_{mn}x_{mn}\Big\|_{L^2(\Omega;X)} \leq C\Big\|\sum_{n=1}^N\sum_{m=1}^M r'_n r_m x_{mn}\Big\|_{L^2(\Omega;X)}.$$

Property (α^-) is defined by the opposite inequality. Properties (α^+) and (α^-) are introduced in [31]. These properties are one sided versions of Pisier's property (α) (see [34]). A space X has property (α) if and only if it has properties (α^+) and (α^-). If X is a Banach function space with finite cotype, then X automatically has property (α). Also note that the Schatten class $S^p(\ell^2)$ has property (α^+) (resp. (α^-) if and only if $p \in [2,\infty)$ (resp. $p \in [1,2]$) (cf. [31] and references therein).

Corollary 4.3. *Let X and Y be Banach spaces and let H be a separable Hilbert space. Let $S : (0,T) \to \mathcal{L}(X,Y)$ be a strongly continuous map. If Y has property (α^+) and $\mathcal{S}$ is semi-R-bounded by some constant K, then for all $B \in \gamma(H,X)$ and all $f \in L^2(0,T)$,*

$$\|fSB\|_{\gamma(0,T;H,Y)} \lesssim_Y K\|f\|_{L^2(0,T)}\|B\|_{\gamma(H,X)}. \tag{4.2}$$

Proof. In [31, Theorem 3.3] it is shown that under property (α^+) the following embedding holds:

$$\gamma(L^2(\mathbb{R}_+),\gamma(H,Y)) \hookrightarrow \gamma(L^2(\mathbb{R}+;H),Y)). \tag{4.3}$$

By (4.3), Propositions 4.2 and 4.1 it follows that

$$\|fSB\|_{\gamma(0,T;H,Y)} \lesssim_Y \|fSB\|_{\gamma(0,T;\gamma(H,Y))} \leq K\|f\|_{L^2(0,T)}\|B\|_{\gamma(H,Y)}.$$

□

The following result will be important in Section 5.

Corollary 4.4. *Let $(T(t))_{t\in\mathbb{R}_+}$ be a strongly continuous semigroup on a Banach space X with $\|T(t)\| \leq Me^{-\omega t}$ for some $\omega > 0$. Assume X has type $p \in [1,2)$. Let $\alpha = \frac{1}{p} - \frac{1}{2}$. The the following assertions hold:*

1. *For all $x \in (X, D(A))_{\alpha,1}$ and all $f \in L^2(\mathbb{R}_+)$, fTx is in $\gamma(\mathbb{R}_+;X)$.*
2. *Let H be a real separable Hilbert space. If additionally X has property (α^+), then for all $B \in \gamma(H,(X,D(A))_{\alpha,1})$ and $f \in L^2(\mathbb{R}_+)$, $fTB \in \gamma(\mathbb{R}_+;H,X)$.*

Proof. (1): This follows from Corollary 3.4 and Proposition 4.1.
(2): This follows from Corollaries 3.4 and 4.3. □

Remark 4.5.

1. If T is not uniformly exponentially stable, then the result of Corollary 4.4 still holds on finite intervals.
2. If X has type 2, then property (α^+) is not needed in Corollary 4.4. This follows from the embedding $L^2(\mathbb{R}_+;\gamma(H,X)) \hookrightarrow \gamma(\mathbb{R}_+;H,X)$ for spaces X with type 2 (see [30, Theorem 5.1]).

5. Applications to stochastic evolution equations

Let X be a real Banach space and let H be a real separable Hilbert space and let $T \in (0,\infty)$. We recall the stochastic Cauchy problem from [29, Section 7],

$$dU(t) = AU(t)\,dt + BdW_H(t), \quad t \in [0,\infty), \quad U(0) = u_0. \tag{5.1}$$

Here A is the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on X and $B \in \mathcal{L}(H,X)$ is a given bounded operator, and $(W_H(t))_{t\in[0,\infty)}$ is a cylindrical Wiener process. We say that (5.1) has a solution if for all $t \in [0,T]$, $\int_0^t S(t-s)B\,dW_H(s)$ exists in $L^2(\Omega;X)$. For details we refer to [29]. Let us recall from [29] that the stochastic integral exists if and only if $t \mapsto S(t)B \in \gamma(0,T;H,X)$.

The next result gives a sufficient condition for the existence of a solution to (5.1). Moreover, the solution has a version with path-wise continuous trajectories.

Theorem 5.1. *Let X be a Banach space. Let Y be a Banach space which is continuously embedded in X and let $i : Y \to X$ denote this embedding. Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on X. If there exists an $\theta \in [0,\frac{1}{2})$ such that*

$$R_{\mathrm{s}}\{t^\theta S(t)i \in \mathcal{L}(Y,X) : t \in [0,T]\} < \infty, \tag{5.2}$$

then the following assertions hold:

(1) *If $B \in \mathcal{L}(H,Y)$ has finite rank, then the problem* (5.1) *has a solution $(U(t))_{t\in[0,T]}$ with continuous paths.*
(2) *If Y has property (α^+) and $B \in \gamma(H,Y)$, then the problem* (5.1) *has a solution $(U(t))_{t\in[0,T]}$ with continuous paths.*

Before we turn to the proof of Theorem 5.1 we give examples for the space Y.

Example 5.2. The semi-R-boundedness assumption (5.2) is fulfilled in the following three cases:

(1) If X has type 2, then (5.2) holds with $Y = X$ and for all $\theta \in [0,\frac{1}{2})$. This follows from Proposition 2.2.
(2) Let $p \in [1,2)$. If X has type p, then (5.2) holds with $Y = (X, D(A))_{\frac{1}{p}-\frac{1}{2},1}$ and for all $\theta \in [0,\frac{1}{2})$. This follows from Corollary 3.4.
(3) If S is analytic, then (5.2) holds with $Y = X$ and $\theta \in [0,\frac{1}{2})$. This follows from the fact that $\frac{d}{dt}[t^\theta S(t)x] \in L^1(0,T;X)$ for all $x \in X$ and [38, Proposition 2.5].

Proof of Theorem 5.1. (1): We can write $B = \sum_{n=1}^N h_n \otimes x_n$, where $(h_n)_{n=1}^N$ are orthonormal and $(x_n)_{n=1}^N$ are in Y. It follows from Proposition 4.1 that for all $\varepsilon \in (0, 1/2-\theta)$,

$$\begin{aligned}\|t \mapsto t^{-\varepsilon}S(t)B\|_{\gamma(0,T;H,X)} &\leq \sum_{n=1}^N \|t \mapsto t^{-\varepsilon}S(t)x_n\|_{\gamma(0,T;H,X)} \\ &\leq K_{S,\theta,T}\sum_{n=1}^N \|x_n\|\,\|t \mapsto t^{-\varepsilon-\theta}\|_{L^2(0,T)} < \infty.\end{aligned}$$

By the Banach space version of the factorization method of [8] (cf. [27] or [37]), this implies that there exists a solution with continuous paths.

(2): This follows as in (1), but this time using Corollary 4.3. □

Note that the family S itself in Example 5.2 (3) does not have to be semi-R-bounded. This follows from the following counterexample.

Example 5.3. Let $X = L^1(0,1)$ and consider $A = \frac{d^2}{dx^2}$, with $D(A) = \{x \in W^{2,1}(0,1) : x(0) = x(1) = 0\}$. Then A generates an analytic semigroup $(S(t))_{t\geq 0}$. It follows from Theorem 5.1 and Example 5.2 that (5.1) has a solution with continuous paths. However, $(S(t))_{t\in[0,1]}$ is not semi-R-bounded. Indeed, it follows from the results in [14] that there do not exist semi-R-bounded semigroups in $L^1(0,1)$ with the property that every $S(t)$ for $t > 0$ is weakly compact. Since it is well-known that $S(t)$ is compact for all $t > 0$ the result follows.

6. Applications to Schauder decompositions

Let X be a Banach space. A sequence of bounded linear operator $(D_n)_{n\geq 1}$ in $\mathcal{L}(X)$ is called a *Schauder decomposition of* X if $D_nD_m = 0$ for $n \neq m$ and for all $x \in X$, one has $x = \sum_{n\geq 1} D_n x$. The corresponding *partial sum projections* $(P_n)_{n\geq 1}$ are defined by $P_n = \sum_{k=1}^n D_k$. The Schauder decomposition $(D_n)_{n\geq 1}$ is called *unconditional* if $\sum_{n\geq 1} D_n x$ converges unconditionally for all $x \in X$. Under geometric conditions (property (Δ) or weak-(α)) on the space X it was shown in [32] that $(P_n)_{n\geq 1}$ is R-bounded. Below we show that without any geometric assumption one always has that $(P_n)_{n\geq 1}$ is semi-R-bounded.

Theorem 6.1. *Let X be a Banach space. If $(D_n)_{n\geq 1}$ is an unconditional Schauder decomposition then the corresponding partial sum projections $(P_n)_{n\geq 1}$ are semi-R-bounded.*

As a consequence it follows that the column and row projections in $\mathcal{S}^1$ are at least semi-R-bounded. In [32] it is shown that they are not R-bounded.

For the proof we need a vector-valued Stein inequality for martingales with independent and symmetric increments.

Lemma 6.2. *Let $(S, \mathcal{F}, \mu)$ be a probability space. Let $\mathcal{I} \subset \mathbb{R}^+$ be an index set which starts at 0. Let $p \in [1,\infty)$ and let F be the set of all $f \in L^p(S;X)$ such that $(\mathbb{E}(f|\mathcal{F}_t))_{t\in\mathcal{I}}$ defines a martingale that starts at zero and which has symmetric and independent increments. For $t \in \mathcal{I}$ let $\mathbb{E}_{\mathcal{F}_t} \in \mathcal{L}(F)$ be defined by $\mathbb{E}_{\mathcal{F}_t} f = \mathbb{E}(f|\mathcal{F}_t)$, then for all $f \in F$ and all choices $t_1, \ldots, t_N \in \mathcal{I}$ and $a_1, \ldots, a_N \in \mathbb{K}$, one has*

$$\mathbb{E}\Big\| \sum_{n=1}^N r_n a_n \mathbb{E}_{\mathcal{F}_{t_n}} f\Big\|^2_{L^p(S;X)} \leq 8 \sum_{n=1}^N |a_n|^2 \|f\|^2_{L^p(S;X)}.$$

Remark 6.3.

(1) If X has property (Δ), then $\{\mathbb{E}_{\mathcal{F}_t}\}_{t\in\mathcal{I}}$ in $\mathcal{L}(F)$ is R-bounded (see [28, Lemma 2.8]).
(2) If X is a UMD space and $p \in (1,\infty)$, then $\{\mathbb{E}_{\mathcal{F}_t}\}_{t\in\mathcal{I}}$ in $\mathcal{L}(L^p(S;X))$ is R-bounded (see [5], [6, Proposition 3.8]). It is not known whether this is true for a wider class than UMD spaces.

Proof of Lemma 6.2. We can assume $t_1 \leq t_2, \ldots \leq t_N$ and let $t_0 = 0$. For $1 \leq n \leq N$, write $d_n = \mathbb{E}_{\mathcal{F}_{t_n}} f - \mathbb{E}_{\mathcal{F}_{t_{n-1}}} f$. Then $(d_n)_{n=1}^N$ are independent and symmetric. Expectation with respect to $(r_n)_{n\geq 1}$ will be denoted with $\mathbb{E}_r$. We have

$$\sum_{n=1}^N r_n \mathbb{E}_{\mathcal{F}_{t_n}} a_n f = \sum_{n=1}^N \sum_{m=1}^n a_n r_n d_m = \sum_{m=1}^N d_m \sum_{n=m}^N a_n r_n.$$

By the Kahane contraction principle applied to $(d_m)_{m=1}^N$ and the Lévy-Octaviani inequalities (cf. [24]) applied to $(a_n r_n)_{n\leq N}$ it follows that

$$\begin{aligned}\mathbb{E}_r\Big\|\sum_{n=1}^N a_n r_n \mathbb{E}_{\mathcal{F}_{t_n}} f\Big\|^2_{L^p(S;X)} &= \mathbb{E}_r\Big\|\sum_{m=1}^N d_m \sum_{n=m}^N a_n r_n\Big\|^2_{L^p(S;X)}\\ &\leq 4\mathbb{E}_r \sup_{1\leq m\leq N}\Big|\sum_{n=m}^N a_n r_n\Big|^2 \Big\|\sum_{m=1}^N d_m\Big\|^2_{L^p(S;X)}\\ &\leq 8\sum_{n=1}^N |a_n|^2 \|f\|^2_{L^p(S;X)}.\end{aligned}$$

□

Proof of Theorem 6.1. We follow the arguments in [32, Corollary 6.2]. Let $(\tilde r_n)_{n\geq 1}$ be a Rademacher sequence on some probability space $(S, \mathcal{F}, \mu)$.

Define $g : X \to L^2(S;X)$ by $g(x) = \sum_{n=1}^\infty \tilde r_n D_n x$. Then g is well-defined and for all $x \in X$

$$(C^-)^{-1}\|x\| \leq \|g(x)\|_{L^2(S;X)} \leq C^+\|x\|, \text{ where } C^-, C^+ > 0 \text{ are constants.} \tag{6.1}$$

For $n \geq 0$ let $\mathcal{F}_n = \sigma(\tilde r_1, \ldots, \tilde r_n)$. It is clear that

$$\tilde{\mathbb{E}}(g(x)|\mathcal{F}_n) = g(P_n x), \quad x \in X,\ n \geq 1.$$

Furthermore, the martingale $(\tilde{\mathbb{E}}(g(x)|\mathcal{F}_n))_{n\geq 0}$ starts at zero and has independent increments. By Lemma 6.2 and (6.1) we obtain that for all $x \in X$ and $a_1, \ldots, a_N \in \mathbb{K}$,

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n P_n a_n x\Big\|^2\Big)^{\frac12} &\leq C^-\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n g(P_n a_n x)\Big\|^2_{L^2(S;X)}\Big)^{\frac12}\\ &= C^-\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n a_n \tilde{\mathbb{E}}(g(x)|\mathcal{F}_n)\Big\|^2_{L^2(S;X)}\Big)^{\frac12}\\ &\leq 2\sqrt{2}C^-\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n a_n g(x)\Big\|^2_{L^2(S;X)}\Big)^{\frac12}\\ &\leq 2\sqrt{2}C^-C^+\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n a_n x\Big\|^2_X\Big)^{\frac12}.\end{aligned}$$

□

7. Applications to the H^∞-calculus

Let X be a Banach space. For details on the H^∞-calculus we refer the reader to [13, 21, 23]. We briefly recall the definition here.

For $\sigma \in [0, \pi)$, let $\Sigma_\sigma = \{\lambda \in \mathbb{C} : \arg(\lambda) < \sigma, \lambda \neq 0\}$. As usual $\partial\Sigma_\sigma$ will be orientated counterclockwise. Let $H^\infty(\Sigma_\sigma)$ denote the space of bounded analytic functions $f : \Sigma_\sigma \to \mathbb{C}$ with norm $\|f\|_{H^\infty(\Sigma_\sigma)} = \sup_{\lambda\in\Sigma_\sigma} |f(\lambda)|$. Let

$$H_0^\infty(\Sigma_\sigma) = \Big\{f \in H^\infty(\Sigma_\sigma) : \exists\epsilon > 0 \text{ s.t. } |f(\lambda)| \leq \frac{|z|^\varepsilon}{(1 + |z|^2)^\varepsilon}\Big\}.$$

We say that a closed densely defined operator A on a Banach space X is a *sectorial operator of type* $w \in [0, \pi)$ if A is one to one with dense range, and for all $\sigma \in (w, \pi)$ and for all $\lambda \in \Sigma_\sigma$, $\|\lambda R(\lambda, A)\| \leq C_\sigma$.

Let A be a sectorial operator of type $w \in [0, \pi)$ and fix $\sigma \in (w, \pi)$ and $\nu \in (w, \sigma)$. For $f \in H_0^\infty(\Sigma_\sigma)$ we can define

$$f(A) = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} f(\lambda) R(\lambda, A) \, d\lambda,$$

where the integral converges in the Bochner sense. We say that A has a bounded $H^\infty(\Sigma_\sigma)$-calculus if there is a constant C such that

$$\|f(A)\| \leq C \|f\|_{H^\infty(\Sigma_\sigma)} \quad \text{for all } f \in H_0^\infty(\Sigma_\sigma). \tag{7.1}$$

In this case (7.1) has a unique continuous extension to all $f \in H^\infty(\Sigma_\sigma)$.

Recall the following result:

Theorem 7.1 ([26])**.** *A sectorial operator A on a Hilbert space X has a bounded H^∞-calculus if and only if it has bounded imaginary powers.*

This result does not extend to the Banach space setting (see [7, Section 5]) in the sense that there exist sectorial operators with bounded imaginary powers which do not have an H^∞-calculus.

If one replaces the assumption that A has bounded imaginary powers by the stronger assumption that A has R-bounded imaginary powers, then this implies again that A has a bounded H^∞-calculus. This is proved in [22] (also see [23, Corollary 12.11]). For spaces X with property (α) the boundedness of the H^∞-calculus is characterized by R-bounded imaginary powers (see [22, 23]). Below we prove a characterization of the boundedness of the H^∞-calculus in terms of semi-R-bounded imaginary powers. Here we do not assume that X has property (α), but require that X has nontrivial type. There are many spaces (i.e. $S^p(\ell^2)$ with $p \in (1, \infty)$) which have nontrivial type but fail property (α).

Theorem 7.2. *Let X be a Banach space. Let A be a sectorial operator of type w. Let $\sigma_1, \sigma_2, \sigma_3 \in [w, \pi)$. Consider the following assertions.*

(a) *A has a bounded $H^\infty(\Sigma_{\sigma_1})$-calculus.*
(b) *The families*
$$\mathcal{T}_1 = \{f(A) : \|f\|_{H^\infty(\Sigma_{\sigma_2})} \leq 1\} \subset \mathcal{L}(X)$$
and $\mathcal{T}_1^* \subset \mathcal{L}(X^*)$ are both semi-R-bounded.
(c) *The families*
$$\mathcal{T}_2 = \{e^{-\sigma_3 |t|} A^{it} : t \in \mathbb{R}\} \subset \mathcal{L}(X)$$
and $\mathcal{T}_2^* \subset \mathcal{L}(X^*)$ are both semi-R-bounded.

The following implications hold:

(1) *Assume X has non-trivial type. If $\sigma_2 > \sigma_1$, then (a) $\Rightarrow$ (b).*
(2) *If $\sigma_3 \geq \sigma_2$, then (b) $\Rightarrow$ (c).*
(3) *If $\sigma_1 > \sigma_3$, then (c) $\Rightarrow$ (a).*

Roughly, the theorem can be rephrased as follows: If X has nontrivial type, then the boundedness of the H^∞-calculus is equivalent to semi-R-bounded imaginary powers.

Proof. (1): Fix $\nu \in (\sigma_1, \sigma_2)$. Fix $f \in H_0^\infty(\Sigma_{\sigma_2})$. Note that by [21, Proposition 4.2] or [23, Lemma 12.4]

$$f(A) = \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} f(\lambda)R(\lambda, A)\,d\lambda = \frac{1}{2\pi i}\int_{\partial\Sigma_\nu} f(\lambda)\lambda^{-\frac{1}{2}}A^{\frac{1}{2}}R(\lambda, A)\,d\lambda. \tag{7.2}$$

Let $\varphi_b \in H_0^\infty(\Sigma_{\sigma_2})$ for $b \in \{-1, 1\}$ be defined by $\varphi_b(z) = z^{\frac{1}{2}}(e^{bi\nu} - z)^{-1}$. Then by (7.2) we obtain that

$$\begin{aligned} f(A) &= \lim_{K\to\infty}\sum_{b\in\{-1,1\}}\frac{-be^{bi\nu}}{2\pi i}\sum_{k=-K}^{K}\int_{2^k}^{2^{k+1}} f(e^{bi\nu}t)\varphi_b(t^{-1}A)\,\frac{dt}{t} \\ &= \lim_{K\to\infty}\sum_{b\in\{-1,1\}}\frac{-be^{bi\nu}}{2\pi i}\sum_{k=-K}^{K}\int_{1}^{2} f(2^ke^{bi\nu}t)\varphi_b(2^{-k}t^{-1}A)\,\frac{dt}{t} \qquad (7.3)\\ &= \lim_{K\to\infty}\sum_{b\in\{-1,1\}}\frac{-be^{bi\nu}}{2\pi i}\int_{1}^{2}\sum_{k=-K}^{K} f(2^ke^{bi\nu}t)\varphi_b(2^{-k}t^{-1}A)\,\frac{dt}{t}. \end{aligned}$$

Using this we show the semi-R-boundedness of $\mathcal{T}_1$. Let $x \in X$ be arbitrary. Let $a_1, \ldots, a_N \in \mathbb{K}$ be arbitrary. By Remark 2.4 (2) it suffices to consider $f_1, \ldots, f_N \in H_0^\infty(\Sigma_{\sigma_2})$ with $\|f_n\|_{H^\infty(\Sigma_{\sigma_2})} \le 1$, $n = 1, \ldots, N$. Fix $\omega \in \Omega$. Let $x^* \in X^*$ be such that $\|x^*\| \le 1$ and

$$\Big\|\sum_{n=1}^{N} r_n(\omega)a_nf_n(A)x\Big\| = \Big\langle\sum_{n=1}^{N} r_n(\omega)a_nf_n(A)x, x^*\Big\rangle.$$

Then with $F_k : [1, 2] \times \Omega \to X$ given by $F_k(t, \omega) = \sum_{n=1}^{N} r_n(\omega)a_nf_n(2^ke^{bi\nu}t)$, it follows from (7.3) that

$$\begin{aligned} &\Big\|\sum_{n=1}^{N} r_n(\omega)a_nf_n(A)x\Big\| \\ &= \lim_{K\to\infty}\sum_{b\in\{-1,1\}}\frac{-be^{bi\nu}}{2\pi i}\int_1^2\sum_{n=1}^{N} r_n(\omega)a_n\sum_{k=-K}^{K}\langle f_n(2^ke^{bi\nu}t)\varphi_b(2^{-k}t^{-1}A)x, x^*\rangle\,\frac{dt}{t} \\ &= \lim_{K\to\infty}\sum_{b\in\{-1,1\}}\frac{-be^{bi\nu}}{2\pi i}\int_1^2\sum_{k=-K}^{K}\langle F_k(t,\omega)\varphi_b(2^{-k}t^{-1}A)x, x^*\rangle\,\frac{dt}{t} \end{aligned}$$

Let $(\tilde r_k)_{k\in\mathbb{Z}}$ be a Rademacher sequence on some probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Expectations with respect to $\tilde\Omega$ and Ω will be denoted by $\tilde{\mathbb{E}}$ and $\mathbb{E}$ respectively. It follows that

$$\begin{aligned} &\Big|\sum_{k=-K}^{K}\langle F_k(t,\omega)\varphi_b(2^{-k}t^{-1}A)x, x^*\rangle\Big| \\ &= \Big|\tilde{\mathbb{E}}\Big\langle\sum_{k=-K}^{K}\tilde r_kF_k(t,\omega)\varphi_b^{\frac{1}{2}}(2^{-k}t^{-1}A)x, \sum_{k=-K}^{K}\tilde r_k\varphi_b^{\frac{1}{2}}(2^{-k}t^{-1}A)^*x^*\Big\rangle\Big| \\ &\le \Big(\tilde{\mathbb{E}}\Big\|\sum_{k=-K}^{K}\tilde r_kF_k(t,\omega)\varphi_b^{\frac{1}{2}}(2^{-k}t^{-1}A)x\Big\|^2\Big)^{\frac{1}{2}}\Big(\tilde{\mathbb{E}}\Big\|\sum_{k=-K}^{K}\tilde r_k\varphi_b^{\frac{1}{2}}(2^{-k}t^{-1}A)^*x^*\Big\|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Recall from [23, Theorem 12.2] that

$$\tilde{\mathbb{E}}\Big\|\sum_{k=-K}^{K}\tilde{r}_k\varphi_b^{\frac12}(2^{-k}t^{-1}A)x\Big\|\leq C_1\|x\|, \tag{7.4}$$

$$\tilde{\mathbb{E}}\Big\|\sum_{k=-K}^{K}\tilde{r}_k\varphi_b^{\frac12}(2^{-k}t^{-1}A)^*x^*\Big\|\leq C_2\|x^*\|. \tag{7.5}$$

By integration over Ω, Fatou's lemma, the Kahane-Khintchine inequality and (7.5) we can conclude that

$$\mathbb{E}\Big\|\sum_{n=1}^{N} r_n a_n f_n(A)x\Big\| \quad \lesssim C_2\liminf_{K\to\infty}\int_1^2 \mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{k=-K}^{K}\tilde{r}_k F_k(t,\cdot)\varphi_b^{\frac12}(2^{-k}t^{-1}A)x\Big\|\,\frac{dt}{t}.$$

Since X has non-trivial type, it also has some cotype $q<\infty$ (see [9, Chapter 13]). Therefore, by [16, Lemma 3.1] or [20, Lemma 3.1] we obtain that

$$\begin{aligned}
&\tilde{\mathbb{E}}\mathbb{E}\Big\|\sum_{k=-K}^{K}\tilde{r}_k F_k(t)\varphi_b^{\frac12}(2^{-k}t^{-1}A)x\Big\| \\
&\quad\lesssim_{X,q}\sup_{k\in\mathbb{Z}}\|F_k(t,\cdot))\|_{L^{q+1}(\Omega)}\Big(\tilde{\mathbb{E}}\Big\|\sum_{k=-K}^{K}\tilde{r}_k\varphi_b^{\frac12}(2^{-k}t^{-1}A)x\Big\|^2\Big)^{\frac12} \\
&\quad\leq\sup_{k\in\mathbb{Z}}\|F_k(t,\cdot)\|_{L^{q+1}(\Omega)}C_1\|x\|,
\end{aligned}$$

where in the last line we used (7.4) and the Kahane-Khintchine inequality. Again by the Khintchine inequality it follows that

$$\|F_k(t,\cdot)\|_{L^{q+1}(\Omega)}\lesssim_q\|F_k(t,\cdot)\|_{L^2(\Omega)}=\Big(\sum_{n=1}^{N}|a_n|^2|f_n(2^k e^{bi\nu}t)|^2\Big)^{\frac12}\leq\Big(\sum_{n=1}^{N}|a_n|^2\Big)^{\frac12}.$$

Putting things together we obtain that

$$\mathbb{E}\Big\|\sum_{n=1}^{N}r_n a_n f_n(A)x\Big\|\lesssim_{X,q}C_1C_2\Big(\sum_{n=1}^{N}|a_n|^2\Big)^{\frac12}\|x\|$$

which proves the semi-R-boundedness of $\mathcal{T}_1$.

The semi-R-boundedness of $\mathcal{T}_1^*$ can be proved in a similar way. Indeed, taking adjoints in (7.3) one obtains that

$$f(A)^*=\lim_{K\to\infty}\sum_{b\in\{-1,1\}}\frac{-be^{bi\nu}}{2\pi i}\int_1^2\sum_{k=-K}^{K}f(2^k e^{bi\nu}t)\varphi_b(2^{-k}t^{-1}A)^*\,\frac{dt}{t}.$$

Fix $x^*\in X^*$. Let $a_1,\ldots,a_N\in\mathbb{K}$ be arbitrary. Let $f_1,\ldots,f_N\in H_0^\infty(\Sigma_{\sigma_2})$ be such that $\|f_n\|_{H^\infty(\sigma_2)}\leq 1$, $n=1,\ldots,N$. Fix $\delta>0$. Fix $\omega\in\Omega$. Let $x\in X$ be such that $\|x\|\leq 1+\delta$ and

$$\Big\|\sum_{n=1}^{N}r_n(\omega)a_n f_n(A)^*x^*\Big\|=\Big\langle x,\sum_{n=1}^{N}r_n(\omega)a_n f_n(A)^*x^*\Big\rangle.$$

Then it follows that

$$\Big\|\sum_{n=1}^{N}r_n(\omega)a_n f_n(A)^*x^*\Big\|$$

$$= \lim_{K\to\infty} \sum_{b\in\{-1,1\}} \frac{-be^{bi\nu}}{2\pi i} \int_1^2 \sum_{k=-K}^{K} \langle x, F_k(t,\omega)\varphi_b(2^{-k}t^{-1}A)^* x^*\rangle \frac{dt}{t}.$$

Then applying (7.4) instead of (7.5) in the same way as before, we obtain that

$$\mathbb{E}\Big\| \sum_{n=1}^{N} r_n a_n f_n(A)^* x^* \Big\|$$
$$\lesssim (1+\delta) C_1 \liminf_{K\to\infty} \int_1^2 \mathbb{E}\tilde{\mathbb{E}}\Big\| \sum_{k=-K}^{K} \tilde{r}_k F_k(t)\varphi_b^{\frac{1}{2}}(2^{-k}t^{-1}A)^* x^* \Big\| \frac{dt}{t}.$$

Since X has non-trivial type, X^* has finite cotype. Therefore, one can complete the proof in the same way as before.

(2): This follows by taking $f_t(z) = e^{-\sigma_3|t|}z^{it}$, $t \in \mathbb{R}$.

(3): It follows from Proposition 4.1 that

$$\begin{aligned}\|t \mapsto e^{-\sigma_1|t|}A^{it}x\|_{\gamma(\mathbb{R};X)} &= \|t \mapsto e^{-(\sigma_1-\sigma_3)|t|}e^{-\sigma_2|t|}A^{it}x\|_{\gamma(\mathbb{R};X)} \\ &\le C\|e^{-(\sigma_1-\sigma_3)|\cdot|}\|_{L^2(\mathbb{R})}\|x\| \lesssim_{\sigma_1,\sigma_3} C\|x\|.\end{aligned}$$

In the same way we obtain that

$$\|t \mapsto e^{-\sigma_1|t|}(A^{it})^* x^*\|_{\gamma(\mathbb{R};X)} \lesssim_{\sigma_1,\sigma_3} C\|x^*\|.$$

Now the result follows from [22, Theorem 7.2]. □

Corollary 7.3. *Let X be a Banach space with property (α). Let A be a sectorial operator of type w and let $\sigma > w$. If the families*

$$\mathcal{T}(\sigma) = \{f(A) : \|f\|_{H^\infty(\Sigma_\sigma)} \le 1\} \subset \mathcal{L}(X)$$

and $\mathcal{T}^*(\sigma) \subset \mathcal{L}(X^*)$ are both semi-R-bounded, then $\mathcal{T}(\sigma')$ is R-bounded for all $\sigma' > \sigma$.

Proof. This follows from Theorem 7.2, and [22, Corollary 7.5] or [23, Theorem 12.8]. □

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