EMBEDDING VECTOR-VALUED BESOV SPACES INTO SPACES OF γ-RADONIFYING OPERATORS

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Abstract. It is shown that a Banach space E has type p if and only for some (all) $d \geqslant 1$ the Besov space $B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;E)$ embeds into the space $\gamma(L^2(\mathbb{R}^d),E)$ of γ-radonifying operators $L^2(\mathbb{R}^d) \to E$. A similar result characterizing cotype q is obtained. These results may be viewed as E-valued extensions of the classical Sobolev embedding theorems.

1. Introduction

Let E be a real or complex Banach space and denote by $\mathscr{S}(\mathbb{R}^d;E)$ the Schwartz space of smooth, rapidly decreasing functions $f : \mathbb{R}^d \to E$. For a function $f \in \mathscr{S}(\mathbb{R}^d;E)$ we consider the linear mapping $I_f : L^2(\mathbb{R}^d) \to E$ defined by

$$I_f g = \int_{\mathbb{R}^d} f(x)g(x)\,dx.$$

The aim of this paper is to prove the following characterization of Banach spaces E with type p in terms of the embeddability of certain E-valued Besov spaces into spaces of γ-radonifying operators with values in E and vice versa. The precise definitions of the spaces $B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;E)$ and $\gamma(L^2(\mathbb{R}^d),E)$ are recalled below.

Theorem 1.1. *Let E be a Banach space and let $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$.*

(1) *E has type p if and only if for some (all) $d \geqslant 1$ the mapping $I : f \mapsto I_f$ extends to a continuous embedding*

$$B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;E) \hookrightarrow \gamma(L^2(\mathbb{R}^d),E);$$

(2) *E has cotype q if and only if for some (all) $d \geqslant 1$ the mapping $I^{-1} : I_f \mapsto f$ extends to a continuous embedding*

$$\gamma(L^2(\mathbb{R}^d),E) \hookrightarrow B^{(\frac{1}{q}-\frac{1}{2})d}_{q,q}(\mathbb{R}^d;E).$$

A version of this result for bounded open domains in $\mathbb{R}^d$ is obtained as well.

As is well known [8, 19], see also [17], E has type 2 if and only if the mapping $f \mapsto I_f$ extends to a continuous embedding $L^2(\mathbb{R}^d;E) \hookrightarrow \gamma(L^2(\mathbb{R}^d),E)$, and E has cotype 2 if and only if $\gamma(L^2(\mathbb{R}^d),E) \hookrightarrow L^2(\mathbb{R}^d;E)$. Thus in some sense, Theorem 1.1 may be viewed as an extension of these results for general values of p and q.

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If $\dim E = 1$, then $\gamma(L^2(\mathbb{R}^d); E) = L^2(\mathbb{R}^d)$ and the embeddings of Theorem 1.1 reduce to the well-known Sobolev embeddings

$$B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow B^{(\frac{1}{q}-\frac{1}{2})d}_{q,q}(\mathbb{R}^d), \quad 1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty.$$

Vector-valued Besov spaces have attracted recent attention in the theory of parabolic evolution equations in Banach spaces as a tool for establishing optimal regularity results; see for instance [1, 4]. In [7], Fourier multiplier theorems with optimal exponents are established for operator-valued multipliers on Besov spaces of functions taking values in Banach spaces with Fourier type p.

On the other hand, the spaces $\gamma(L^2(\mathbb{R}^d), E)$ have recently played an important role in the theory of H^∞-functional calculus for sectorial operators [6, 10, 11] and the theory of wavelet decompositions [9]. Furthermore, the spaces $\gamma(L^2(\mathbb{R}^d), E)$ have been characterized in terms of stochastic integrals with respect to (cylindrical) Brownian motions [15, 16, 19]. Therefore, our results allow to compare various square functions and they also give conditions for the stochastic integrability of E-valued functions. These applications, which motivated our results, will be detailed in a forthcoming paper.

Throughout this paper, H is a Hilbert space and E is a Banach space, which may be taken both real or both complex. Furthermore, $(r_n)_{n\geqslant 1}$ denotes a Rademacher sequence and $(\gamma_n)_{n\geqslant 1}$ a Gaussian sequence.

1.1. **Type and cotype.** Let $p \in [1, 2]$ and $q \in [2, \infty]$. A Banach space E is said to have *type* p if there exists a constant $C \geqslant 0$ such that for all finite subsets $\{x_1, \dots, x_N\}$ of E we have

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n x_n\Big\|^2\Big)^{\frac{1}{2}} \leqslant C\Big(\sum_{n=1}^N \|x_n\|^p\Big)^{\frac{1}{p}}.$$

The least possible constant C is called the *type p constant* of E and is denoted by $T_p(E)$. A Banach space E is said to have *cotype* q if there exists a constant $C \geqslant 0$ such that for all finite subsets $\{x_1, \dots, x_N\}$ of E we have

$$\Big(\sum_{n=1}^N \|x_n\|^q\Big)^{\frac{1}{q}} \leqslant C\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n x_n\Big\|^2\Big)^{\frac{1}{2}},$$

with the obvious modification in the case $q = \infty$. The least possible constant C is called the *cotype q constant* of E and is denoted by $C_p(E)$. As is well known, in both definitions the rôle of the Rademacher variables may be replaced by Gaussian variables without altering the class of spaces under consideration. The least constants arising from these equivalent definitions are called the *Gaussian type p constant* and the *Gaussian cotype q constant* of E respectively, notation $T^\gamma_p(E)$ and $C^\gamma_q(E)$.

Every Banach space has type 1 and cotype ∞. The L^p-spaces have type $\min\{p, 2\}$ and cotype $\max\{p, 2\}$ for $1 \leqslant p < \infty$. Every Hilbert space has both type 2 and cotype 2, and a famous result of Kwapień asserts that up to isomorphism this property characterizes the class of Hilbert spaces.

For more information we refer to Maurey's survey article [12] and the references given therein.

1.2. **Besov spaces.** Next we recall the definition of Besov spaces using the so-called Littlewood-Paley decomposition. We follow the approach of Peetre; see [21, Section 2.3.2] (where the scalar-valued case is considered) and [1, 7, 20]. The Fourier transform of a function $f \in L^1(\mathbb{R}^d; E)$ will be normalized as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^d.$$

Let $\phi \in \mathscr{S}(\mathbb{R}^d)$ be a fixed Schwartz function whose Fourier transform $\widehat{\phi}$ is non-negative and has support in $\{\xi \in \mathbb{R}^d : \ \frac{1}{2} \leqslant |\xi| \leqslant 2\}$ and which satisfies

$$\sum_{k\in\mathbb{Z}} \widehat{\phi}(2^{-k}\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

Define the sequence $(\varphi_k)_{k\geqslant 0}$ in $\mathscr{S}(\mathbb{R}^d)$ by

$$\widehat{\varphi_k}(\xi) = \widehat{\phi}(2^{-k}\xi) \quad \text{for} \quad k = 1, 2, \dots \quad \text{and} \quad \widehat{\varphi_0}(\xi) = 1 - \sum_{k\geqslant 1} \widehat{\varphi_k}(\xi), \quad \xi \in \mathbb{R}^d.$$

For $1 \leqslant p, q \leqslant \infty$ and $s \in \mathbb{R}$ the *Besov space* $B^s_{p,q}(\mathbb{R}^d; E)$ is defined as the space of all E-valued tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d; E)$ for which

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;E)} := \Big\| \big(2^{ks}\varphi_k * f\big)_{k\geqslant 0} \Big\|_{l^q(L^p(\mathbb{R}^d;E))}$$

is finite. Endowed with this norm, $B^s_{p,q}(\mathbb{R}^d; E)$ is a Banach space, and up to an equivalent norm this space is independent of the choice of the initial function ϕ. The sequence $(\varphi_k * f)_{k\geqslant 0}$ is called the *Littlewood-Paley decomposition* of f associated with the function ϕ.

The following continuous inclusions hold:

$$B^s_{p,q_1}(\mathbb{R}^d; E) \hookrightarrow B^s_{p,q_2}(\mathbb{R}^d; E), \ B^{s_1}_{p,q}(\mathbb{R}^d; E) \hookrightarrow B^{s_2}_{p,q}(\mathbb{R}^d; E)$$

for all $s, s_1, s_2 \in \mathbb{R}$, $p, q, q_1, q_2 \in [1, \infty]$ with $q_1 \leqslant q_2$, $s_2 \leqslant s_1$. Also note that

$$B^0_{p,1}(\mathbb{R}^d; E) \hookrightarrow L^p(\mathbb{R}^d; E) \hookrightarrow B^0_{p,\infty}(\mathbb{R}^d; E).$$

If $1 \leqslant p, q < \infty$, then $B^s_{p,q}(\mathbb{R}^d; E)$ contains the Schwartz space $\mathscr{S}(\mathbb{R}^d; E)$ as a dense subspace.

In Section 3 we shall need the following lemma. For $\lambda > 0$ let $f_\lambda(x) := f(\lambda x)$.

Lemma 1.2. *Let $p, q \in [1, \infty]$ and $s \in \mathbb{R}$, $s \neq 0$.*

(1) *If $s > 0$, there exists a constant $C > 0$ such that for all $\lambda = 2^n$, $n \geqslant 1$, and $f \in B^s_{p,q}(\mathbb{R}^d; E)$ we have*

$$\|f_\lambda\|_{B^s_{p,q}(\mathbb{R}^d;E)} \leqslant C\lambda^{s-\frac{d}{p}} \|f\|_{B^s_{p,q}(\mathbb{R}^d;E)}.$$

(2) *If $s < 0$, there exists a constant $C > 0$ such that for all $\lambda = 2^n$, $n \leqslant -1$, and $f \in B^s_{p,q}(\mathbb{R}^d; E)$ we have*

$$\|f_\lambda\|_{B^s_{p,q}(\mathbb{R}^d;E)} \leqslant C\lambda^{s-\frac{d}{p}} \|f\|_{B^s_{p,q}(\mathbb{R}^d;E)}.$$

Proof. We only prove (1), the proof of (2) being similar. The proofs are patterned after [22, Proposition 3.4.1].

Let ϕ and φ_k, $k = 0, 1, 2, \dots$, be as in Subsection 1.2. Define, for $m \in \mathbb{Z}$, the functions ψ_m by $\widehat{\psi_m}(\xi) := \widehat{\phi}(2^{-m}\xi)$. Then $\psi_m = \varphi_m$ for $m = 1, 2, \dots$ and $(\widehat{\psi_m})_\lambda = \widehat{\psi_{m-n}}$ for $m \in \mathbb{Z}$ and $\lambda = 2^n$, $n \in \mathbb{Z}$. For $s > 0$ we have

$$\begin{aligned}
\Big(\sum_{k\geqslant 0} 2^{ksq}\|\varphi_k * f_\lambda\|^q_{L^p(\mathbb{R}^d;E)}\Big)^{\frac{1}{q}} &= \lambda^{-\frac{d}{p}}\Big(\sum_{k\geqslant 0} 2^{ksq}\|\mathscr{F}^{-1}((\widehat{\varphi_k})_\lambda \widehat{f})\|^q_{L^p(\mathbb{R}^d;E)}\Big)^{\frac{1}{q}} \\
&\leqslant \lambda^{-\frac{d}{p}}\|\mathscr{F}^{-1}((\widehat{\varphi_0})_\lambda \widehat{f})\|_{L^p(\mathbb{R}^d;E)} \\
&\quad + \lambda^{-\frac{d}{p}}\Big(\sum_{k=1}^{n} 2^{ksq}\|\mathscr{F}^{-1}(\widehat{\psi_{k-n}}\widehat{f})\|^q_{L^p(\mathbb{R}^d;E)}\Big)^{\frac{1}{q}} \\
&\quad + \lambda^{s-\frac{d}{p}}\Big(\sum_{l\geqslant 1} 2^{lsq}\|\mathscr{F}^{-1}(\widehat{\psi_l}\widehat{f})\|^q_{L^p(\mathbb{R}^d;E)}\Big)^{\frac{1}{q}} \\
&=: \mathrm{(I)} + \mathrm{(II)} + \mathrm{(III)}.
\end{aligned}$$

Since $\widehat{\varphi_0} = 1$ on $(0, 1]$ and $(\widehat{\varphi_0})_\lambda$ has support in $(0, 2^{-n}] \subseteq (0, \frac{1}{2}]$, by Young's inequality we have

$$\|\mathscr{F}^{-1}((\widehat{\varphi_0})_\lambda \widehat{f})\|_{L^p(\mathbb{R}^d;E)} = \|\mathscr{F}^{-1}((\widehat{\varphi_0})_\lambda \widehat{\varphi_0}\widehat{f})\|_{L^p(\mathbb{R}^d;E)} \leqslant \|\varphi_0\|_{L^1(\mathbb{R}^d)}\|\varphi_0 * f\|_{L^p(\mathbb{R}^d;E)}.$$

Hence,

$$\mathrm{(I)} \leqslant \lambda^{-\frac{d}{p}}\|\varphi_0\|_{L^1(\mathbb{R}^d)}\|f\|_{B^s_{p,p}(\mathbb{R}^d;E)} \leqslant \lambda^{s-\frac{d}{p}}\|\varphi_0\|_{L^1(\mathbb{R}^d)}\|f\|_{B^s_{p,q}(\mathbb{R}^d;E)}.$$

To estimate (II) we note that for $k = 1, \dots, n-1$ the functions $\widehat{\psi_{k-n}}$ have support in $(0, 1]$. Therefore,

$$\begin{aligned}
\|\mathscr{F}^{-1}(\widehat{\psi_{k-n}}\widehat{f})\|_{L^p(\mathbb{R}^d;E)} &\leqslant \|\psi_{k-n}\|_{L^1(\mathbb{R}^d)}\|\varphi_0 * f\|_{L^p(\mathbb{R}^d;E)} \\
&= \|\phi\|_{L^1(\mathbb{R}^d)}\|\varphi_0 * f\|_{L^p(\mathbb{R}^d;E)}.
\end{aligned}$$

Similarly, for $k = n$,

$$\|\mathscr{F}^{-1}((\widehat{\psi_n})_\lambda \widehat{f})\|_{L^p(\mathbb{R}^d;E)} \leqslant \|\phi\|_{L^1(\mathbb{R}^d)}\big(\|\varphi_0 * f\|_{L^p(\mathbb{R}^d;E)} + \|\varphi_1 * f\|_{L^p(\mathbb{R}^d;E)}\big).$$

Summing these terms and using that $s > 0$ we obtain

$$\mathrm{(II)} \leqslant C_{s,q}\lambda^{s-\frac{d}{p}}\|\varphi\|_{L^1(\mathbb{R}^d)}\|f\|_{B^s_{p,q}(\mathbb{R}^d;E)}$$

with a constant $C_{q,s}$ depending only of q and s. Obviously,

$$\mathrm{(III)} \leqslant \lambda^{s-\frac{d}{p}}\|f\|_{B^s_{p,q}(\mathbb{R}^d;E)}.$$

By putting these estimates together the desired inequality follows. □

1.3. **γ-Radonifying operators.** For a finite rank operator $R : H \to E$ of the form

$$Rh = \sum_{n=1}^{N} [h, h_n]_H\, x_n \tag{1.1}$$

with $h_1, \dots, h_N$ orthonormal in H, we define

$$\|R\|^2_{\gamma(H,E)} := \mathbb{E}\Big\|\sum_{n=1}^{N} \gamma_n R h_n\Big\|^2.$$

Note that $\|R\|_{\gamma(H,E)}$ does not depend on the particular representation of R as in (1.1). The completion of the space of finite rank operators with respect to the norm $\|\cdot\|_{\gamma(H,E)}$ defines a two-sided operator ideal $\gamma(H, E)$ in $\mathscr{L}(H, E)$. If

H is separable, an operator $R \in \mathscr{L}(H,E)$ belongs to $\gamma(H,E)$ if and only if for some (equivalently, for every) orthonormal basis $(h_n)_{n\geqslant 1}$ of H the Gaussian sum $\sum_{n\geqslant 1}\gamma_n R h_n$ converges in $L^2(\Omega;E)$, in which case we have

$$\|R\|^2_{\gamma(H,E)} = \mathbb{E}\Big\|\sum_{n\geqslant 1}\gamma_n R h_n\Big\|^2.$$

We refer to [5, Chapter 12] for more information.

The following elementary convergence result, cf. [15, Proposition 2.4], will be useful. If the $T_1, T_2, \ldots \in \mathscr{L}(H)$ and $T \in \mathscr{L}(H)$ satisfy $\sup_{n\geqslant 1}\|T_n\| < \infty$ and $\lim_{n\to\infty} T^* h = T_n^* h$ for all $h \in H$, then for all $R \in \gamma(H,E)$ we have

$$\lim_{n\to\infty}\|R\circ T_n - R\circ T\|_{\gamma(H,E)} = 0. \tag{1.2}$$

If H_1 and H_2 are Hilbert spaces, then every bounded operator $T : H_1 \to H_2$ induces a bounded operator $\widetilde{T} : \gamma(H_1,E) \to \gamma(H_2,E)$ by the formula

$$\widetilde{T}R := R\circ T^*$$

and we have

$$\|\widetilde{T}\|_{\mathscr{L}(\gamma(H_1,E),\gamma(H_2,E))} \leqslant \|T\|_{\mathscr{L}(H_1,H_2)}. \tag{1.3}$$

This extension procedure is introduced in [11] and will be useful below.

If (S,Σ,μ) is a σ-finite measure space, we denote by $\gamma(S;E)$ the vector space of all strongly μ-measurable functions $f : S \to E$ for which $\langle f, x^*\rangle$ belongs to $L^2(S)$ for all $x^* \in E^*$ and the associated Pettis operator $I_f : L^2(S) \to E$,

$$I_f g = \int_S f g\, d\mu$$

belongs to $\gamma(L^2(S),E)$. We identify functions defining the same operator. An easy approximation argument shows that the simple functions in $\gamma(S;E)$ form a dense subspace of $\gamma(L^2(S),E)$. We shall write

$$\|f\|_{\gamma(S;E)} := \|I_f\|_{\gamma(L^2(S),E)}.$$

2. Embedding results for $\mathbb{R}^d$

The proof of Theorem 1.1 is based on two lemmas.

Lemma 2.1.

(1) *Let E have type $p \in [1,2]$. If $f \in \mathscr{S}(\mathbb{R}^d;E)$ satisfies* $\operatorname{supp}\widehat{f} \subseteq [-\pi,\pi]^d$, *then $f \in \gamma(\mathbb{R}^d;E)$ and*

$$\|f\|_{\gamma(\mathbb{R}^d;E)} \leqslant T_p^\gamma(E)\|f\|_{L^p(\mathbb{R}^d;E)},$$

where $T_p^\gamma(E)$ denotes the Gaussian type p constant of E.

(2) *Let E have cotype $q \in [2,\infty]$. If $f \in \mathscr{S}(\mathbb{R}^d;E)$ satisfies* $\operatorname{supp}\widehat{f} \subseteq [-\pi,\pi]^d$, *then*

$$\|f\|_{\gamma(\mathbb{R}^d;E)} \geqslant C_q^\gamma(E)^{-1}\|f\|_{L^q(\mathbb{R}^d;E)},$$

where $C_q^\gamma(E)$ denotes the Gaussian cotype q constant of E.

Proof. Let $Q := [-\pi,\pi]^d$. We consider the functions $h_n(x) = (2\pi)^{-d/2}e^{in\cdot x}$ with $n \in \mathbb{Z}^d$, $x \in Q$, which define an orthonormal basis for $L^2(Q)$.

(1) Define the bounded operators $I_f : L^2(\mathbb{R}^d) \to E$ and $I_{\widehat{f}} : L^2(\mathbb{R}^d) \to E$ by

$$I_f g := \int_{\mathbb{R}^d} f(x)g(x)\,dx, \quad I_{\widehat{f}} g := \int_{\mathbb{R}^d} \widehat{f}(x)g(x)\,dx.$$

In case E is a real Banach space we consider its complexification in the second definition. By the assumption on the support of $\widehat{f}$ we may identify $I_{\widehat{f}}$ with a bounded operator from $L^2(Q)$ to E of the same norm. Since $I_{\widehat{f}} h_n = f(n)$, for any finite subset $F \subseteq \mathbb{Z}^d$ we have

$$\Big(\mathbb{E}\Big\|\sum_{n\in F}\gamma_n I_{\widehat{f}} h_n\Big\|^2\Big)^{\frac12} = \Big(\mathbb{E}\Big\|\sum_{n\in F}\gamma_n f(n)\Big\|^2\Big)^{\frac12} \leqslant T_p^\gamma(E)\Big(\sum_{n\in F}\|f(n)\|^p\Big)^{\frac1p}.$$

It follows that $I_{\widehat{f}} \in \gamma(L^2(Q),E)$. By the identification made above it follows that $I_{\widehat{f}} \in \gamma(L^2(\mathbb{R}),E)$ and

$$\|I_{\widehat{f}}\|_{\gamma(L^2(\mathbb{R}^d),E)} = \|I_{\widehat{f}}\|_{\gamma(L^2(Q),E)} \leqslant T_p^\gamma(E)\Big(\sum_{n\in\mathbb{Z}^d}\|f(n)\|^p\Big)^{\frac1p}.$$

From (1.3) it follows that

$$\|f\|_{\gamma(\mathbb{R}^d;E)} = \|I_f\|_{\gamma(L^2(\mathbb{R}^d),E)} = \|I_{\widehat{f}}\|_{\gamma(L^2(\mathbb{R}^d),E)} \leqslant T_p^\gamma(E)\Big(\sum_{n\in\mathbb{Z}^d}\|f(n)\|^p\Big)^{\frac1p}.$$

For $t \in R := [0,1]^d$ put $f_t(s) = f(s+t)$. Then $\operatorname{supp}\widehat{f_t} \subseteq Q$ and

$$\|f\|_{\gamma(\mathbb{R}^d;E)} = \|f_t\|_{\gamma(\mathbb{R}^d;E)} \leqslant T_p^\gamma(E)\Big(\sum_{n\in\mathbb{Z}^d}\|f_t(n)\|^p\Big)^{\frac1p}.$$

By raising both sides to the power p and integrating over R we obtain

$$\|f\|_{\gamma(\mathbb{R}^d;E)} \leqslant T_p^\gamma(E)\Big(\int_R\sum_{n\in\mathbb{Z}^d}\|f_t(n)\|^p\,dt\Big)^{\frac1p} = T_p^\gamma(E)\Big(\int_{\mathbb{R}^d}\|f(s)\|^p\,ds\Big)^{\frac1p}.$$

(2) This is proved similarly. Note that by part (1) (with $p = 1$) we have $f \in \gamma(\mathbb{R}^d;E)$. □

Let (S,Σ,μ) be a measure space. For a bounded operator $R : L^2(S) \to E$ and a set $S_0 \in \Sigma$ we define $R|_{S_0} : L^2(S) \to E$ by

$$R|_{S_0} g := R(\mathbf{1}_{S_0} g).$$

Note that if $R \in \gamma(L^2(S),E)$, then $R|_{S_0} \in \gamma(L^2(S),E)$ and

$$\|R|_{S_0}\|_{\gamma(L^2(S),E)} \leqslant \|R\|_{\gamma(L^2(S),E)}$$

by the operator ideal property of $\gamma(L^2(S),E)$.

In the following lemma we use the well known fact that if E has type p (cotype q), then the same is true for the space $L^2(\Omega;E)$ and we have

$$T_p(L^2(\Omega;E)) = T_p(E), \quad C_q(L^2(\Omega;E)) = C_q(E).$$

Lemma 2.2. *Let (S,Σ,μ) be a measure space and let $(S_j)_{j\geqslant 1} \subseteq \Sigma$ be a partition of S.*

(1) *Let E have type $p \in [1,2]$. Then for all $R \in \gamma(L^2(S),E)$ we have*

$$\|R\|_{\gamma(L^2(S),E)} \leqslant T_p(E)\Big(\sum_{j\geqslant 1}\|R|_{S_j}\|^p_{\gamma(L^2(S),E)}\Big)^{\frac{1}{p}}.$$

(2) *Let E have cotype $q \in [2,\infty]$. Then for all $R \in \gamma(L^2(S),E)$ we have*

$$\|R\|_{\gamma(L^2(S),E)} \geqslant C_q(E)^{-1}\Big(\sum_{j\geqslant 1}\|R|_{S_j}\|^q_{\gamma(L^2(S),E)}\Big)^{\frac{1}{q}}.$$

Proof. (1) We may assume that $\mu(S_j) > 0$ for all j. Fixing R, we may also assume that Σ is countably generated. As a result, $L^2(S)$ is separable and we may choose an orthonormal basis $(h_{jk})_{j,k\geqslant 1}$ for $L^2(S)$ in such a way that for each j the sequence $(h_{jk})_{k\geqslant 1}$ is an orthonormal basis for $L^2(S_j)$. Let $(\gamma_{jk})_{j,k\geqslant 1}$ and $(r'_j)_{j\geqslant 1}$ be a doubly-indexed Gaussian sequence and a Rademacher sequence on probability spaces $(\Omega,\mathbb{P})$ and $(\Omega',\mathbb{P}')$, respectively. By a standard randomization argument,

$$\begin{aligned}
\|R\|_{\gamma(L^2(S),E)} &= \Big(\mathbb{E}\Big\|\sum_{j,k\geqslant 1}\gamma_{jk}Rh_{jk}\Big\|^2\Big)^{\frac{1}{2}}\\
&= \Big(\mathbb{E}\Big\|\sum_{j,k\geqslant 1}\gamma_{jk}R|_{S_j}h_{jk}\Big\|^2\Big)^{\frac{1}{2}}\\
&= \Big(\mathbb{E}'\Big\|\sum_{j\geqslant 1}r'_j\sum_{k\geqslant 1}\gamma_{jk}R|_{S_j}h_{jk}\Big\|^2_{L^2(\Omega;E)}\Big)^{\frac{1}{2}}\\
&\leqslant T_p(L^2(\Omega;E))\Big(\sum_{j\geqslant 1}\Big\|\sum_{k\geqslant 1}\gamma_{jk}R|_{S_j}h_{jk}\Big\|^p_{L^2(\Omega;E)}\Big)^{\frac{1}{p}}\\
&= T_p(E)\Big(\sum_{j\geqslant 1}\|R|_{S_j}\|^p_{\gamma(L^2(S),E)}\Big)^{\frac{1}{p}}.
\end{aligned}$$

(2) This is proved similarly. □

We are now prepared for the proof of Theorem 1.1. Recall that the Schwartz functions ϕ and φ_k, $k \geqslant 1$, are defined in Subsection 1.2.

Proof of Theorem 1.1. (1) First we prove the 'only if' part and assume that E has type p. Let $f \in \mathscr{S}(\mathbb{R}^d;E)$ and let $f_k := \varphi_k * f$. Putting $g_k(x) := f_k(2^{-k}x)$ we have $g_k \in \mathscr{S}(\mathbb{R}^d;E)$ and

$$\operatorname{supp}\widehat{g_k} \subseteq \{\xi \in \mathbb{R}^d:\ |\xi| \leqslant 2\} \subseteq [-\pi,\pi]^d.$$

Hence from Lemma 2.1 we obtain $f_k \in \gamma(\mathbb{R}^d;E)$ and

$$\begin{aligned}
\|f_k\|_{\gamma(\mathbb{R}^d;E)} &= 2^{-kd/2}\|g_k\|_{\gamma(\mathbb{R}^d;E)}\\
&\leqslant 2^{-kd/2}T^\gamma_p(E)\|g_k\|_{L^p(\mathbb{R}^d;E)} = 2^{\frac{kd}{p}-\frac{kd}{2}}T^\gamma_p(E)\|f_k\|_{L^p(\mathbb{R}^d;E)}.
\end{aligned}$$

Using the Lemma 2.2, applied to the decompositions $(S_{2k})_{k\in\mathbb{Z}}$ and $(S_{2k+1})_{k\in\mathbb{Z}}$ of $\mathbb{R}^d\setminus\{0\}$, we obtain, for all $n\geqslant m\geqslant 0$,

$$\Big\|\sum_{k=2m}^{2n} f_k\Big\|_{\gamma(\mathbb{R}^d;E)} \leqslant T_p^\gamma(E)T_p(E)\Big(\sum_{j=m}^{n} 2^{(\frac{2jd}{p}-\frac{2jd}{2})p}\|f_{2j}\|^p_{L^p(\mathbb{R}^d;E)}\Big)^{\frac{1}{p}}$$
$$+T_p^\gamma(E)T_p(E)\Big(\sum_{j=m}^{n-1} 2^{(\frac{(2j+1)d}{p}-\frac{(2j+1)d}{2})p}\|f_{2j+1}\|^p_{L^p(\mathbb{R}^d;E)}\Big)^{\frac{1}{p}}.$$

Estimating sums of the form $\sum_{k=2m}^{2n+1}$, $\sum_{k=2m+1}^{2n}$, and $\sum_{k=2m+1}^{2n+1}$ in a similar way, it follows that $f\in\gamma(\mathbb{R}^d;E)$ and

$$\|f\|_{\gamma(\mathbb{R}^d;E)}\leqslant 2T_p^\gamma(E)T_p(E)\|f\|_{B_{p,p}^{(\frac{1}{p}-\frac{1}{2})d}(\mathbb{R}^d;E)}.$$

Since $\mathscr{S}(\mathbb{R}^d;E)$ is dense in $B_{p,p}^{(\frac{1}{p}-\frac{1}{2})d}(\mathbb{R}^d;E)$ it follows that the mapping $f\mapsto I_f$ extends to a bounded operator I from $B_{p,p}^{(\frac{1}{p}-\frac{1}{2})d}(\mathbb{R}^d;E)$ into $\gamma(\mathbb{R}^d;E)$ of norm $\|I\|\leqslant 2T_p^\gamma(E)T_p(E)$. The simple proof that I is injective is left to the reader.

Next we prove the 'if' part. For $n\geqslant 1$, let $\psi_n\in\mathscr{S}(\mathbb{R}^d)$ be defined as

$$\widehat{\psi}_n(\xi)=c2^{-nd/2}\widehat{\phi}(2^{-n}\xi),$$

where $c:=\|\phi\|^{-1}_{L^2(\mathbb{R}^d)}$. Then $(\psi_{3n})_{n\geqslant 1}$ is an orthonormal system in $L^2(\mathbb{R}^d)$. For any finite sequence $(x_n)_{n=1}^N$ in E we then have, with $f:=\sum_{n=1}^N\psi_{3n}\otimes x_n$,

$$\|f\|^2_{\gamma(\mathbb{R}^d;E)}=\mathbb{E}\Big\|\sum_{n=1}^N\gamma_n x_n\Big\|^2.$$

Notice that for $k\geqslant 1$,

$$\|\varphi_k*\varphi_k\|_{L^p(\mathbb{R}^d)}=2^{kd-\frac{1}{p}kd}\|\phi*\phi\|_{L^p(\mathbb{R}^d)}$$

and

$$\|\varphi_{k+1}*\varphi_k\|_{L^p(\mathbb{R}^d)}=2^{kd-\frac{1}{p}kd}\|\varphi_1*\phi\|_{L^p(\mathbb{R}^d)}.$$

Therefore, for $n=1,\dots,N$,

$$\|\varphi_{3n}*f\|_{L^p(\mathbb{R}^d;E)}=c2^{-\frac{3}{2}nd}\|\varphi_{3n}*\varphi_{3n}\|_{L^p(\mathbb{R}^d)}\|x_n\|=c2^{(\frac{1}{2}-\frac{1}{p})3nd}\|\phi*\phi\|_{L^p(\mathbb{R}^d)}\|x_n\|$$

and similarly,

$$\|\varphi_{3n-1}*f\|_{L^p(\mathbb{R}^d;E)}=c2^{(\frac{1}{2}-\frac{1}{p})3nd-(1-\frac{1}{p})d}\|\varphi_1*\phi\|_{L^p(\mathbb{R}^d)}\|x_n\|$$

and

$$\|\varphi_{3n+1}*f\|_{L^p(\mathbb{R}^d;E)}=c2^{(\frac{1}{2}-\frac{1}{p})3nd}\|\varphi_1*\phi\|_{L^p(\mathbb{R}^d)}\|x_n\|.$$

Finally, for $k\geqslant 3N+2$ we have $\varphi_k*f=0$. Summing up, it follows that there exists a constant C, depending only on p,d and ϕ such that

$$\|f\|_{B_{p,p}^{(\frac{1}{p}-\frac{1}{2})d}(\mathbb{R}^d;E)}\leqslant C\Big(\sum_{n=1}^N\|x_n\|^p\Big)^{\frac{1}{p}}.$$

By putting things together we see that E has type p, with Gaussian type p constant $T_p^\gamma(E)\leqslant C\|I\|$, where $I: B_{p,p}^{(\frac{1}{p}-\frac{1}{2})d}(\mathbb{R}^d;E)\hookrightarrow\gamma(\mathbb{R}^d;E)$ is the embedding.

(2) This is proved similarly. □

As a special case of Theorem 1.1, note that for every Banach space E we obtain continuous embeddings

$$B^{\frac{1}{2}d}_{1,1}(\mathbb{R}^d;E) \hookrightarrow \gamma(L^2(\mathbb{R}^d),E) \hookrightarrow B^{-\frac{1}{2}d}_{\infty,\infty}(\mathbb{R}^d;E).$$

As is easily checked by going through the proofs, these embeddings are contractive.

Let $H^{\alpha,p}(\mathbb{R}^d;E)$, with $\alpha \in \mathbb{R}$ and $1 \leqslant p < \infty$, denote the usual E-valued Lebesgue-Bessel potential spaces [3, Section 6.2], [21, Section 2.33]. In [10] the γ-Sobolev spaces $\gamma(H^{\alpha,2}(\mathbb{R}^d),E)$ are introduced and their basic properties are studied. From Theorem 1.1 we obtain the following γ-analogue of the Sobolev embedding theorem.

Corollary 2.3.

(1) *If E has type $p \in [1,2]$, we have continuous embeddings*

$$H^{\alpha,p}(\mathbb{R}^d;E) \hookrightarrow B^{\beta+(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;E) \hookrightarrow \gamma(H^{-\beta,2}(\mathbb{R}^d),E)$$

for all $\alpha,\beta \in \mathbb{R}$ satisfying $\alpha > \beta + (\frac{1}{p}-\frac{1}{2})d$.

(2) *If E has cotype $q \in [2,\infty]$, we have continuous embeddings*

$$\gamma(H^{-\beta,2}(\mathbb{R}^d),E) \hookrightarrow B^{\beta+(\frac{1}{q}-\frac{1}{2})d}_{q,q}(\mathbb{R}^d;E) \hookrightarrow H^{\alpha,q}(\mathbb{R}^d;E)$$

for all $\alpha,\beta \in \mathbb{R}$ satisfying $\alpha < \beta + (\frac{1}{q}-\frac{1}{2})d$.

Remark 2.4. Taking $q=\infty$ in (2) we obtain the embedding $\gamma(H^{-\beta,2}(\mathbb{R}^d),E) \hookrightarrow B^{\beta-\frac{d}{2}}_{\infty,\infty}(\mathbb{R}^d;E)$ as a special case. If $\beta - \frac{d}{2}$ is strictly positive and not an integer, the latter space can be identified, up to an equivalent norm, with the Hölder space $(BUC)^{\beta-\frac{d}{2}}(\mathbb{R}^d;E)$ [1, Equation (5.8)] and we thus obtain a continuous embedding

$$\gamma(H^{-\beta,2}(\mathbb{R}^d);E) \hookrightarrow (BUC)^{\beta-\frac{d}{2}}(\mathbb{R}^d;E). \tag{2.1}$$

Proof. The second embedding in (1) and the first embedding in (2) are immediate from Theorem 1.1 combined with the fact that $(I-\Delta)^{-\beta/2}$ acts as an isomorphism from $B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;E)$ onto $B^{\beta+(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;E)$ [1, Theorem 6.1] and from $\gamma(L^2(\mathbb{R}^d),E)$ onto $\gamma(H^{-\beta,2}(\mathbb{R}^d),E)$. The first embedding in (1) and the second embedding in (2) follow from the E-valued analogues of [3, Theorem 6.2.4]. □

Note that (2) can be combined with the classical Sobolev embedding theorem to yield an inclusion result which is slightly weaker than (2.1).

If we combine Theorem 1.1 with the boundedness of the Fourier transform on $\gamma(L^2(\mathbb{R}^d);E)$ we obtain the following result for the Fourier transform on $\mathbb{R}^d$.

Corollary 2.5. *Let E be a Banach space with type $p \in [1,2]$ and cotype $q \in [2,\infty]$. Then the Fourier transform is a bounded operator from $B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;E)$ into $B^{(\frac{1}{q}-\frac{1}{2})d}_{q,q}(\mathbb{R}^d;E)$.*

3. Embedding results for bounded domains

Let D be a nonempty bounded open domain in $\mathbb{R}^d$. For $1 \leqslant p,q \leqslant \infty$ and $s \in \mathbb{R}$ we define

$$B^s_{p,q}(D;E) = \{f|_D : \ f \in B^s_{p,q}(\mathbb{R}^d;E)\}.$$

This space is a Banach space endowed with the norm

$$\|g\|_{B^s_{p,q}(D;E)} = \inf_{f|_D=g} \|f\|_{B^s_{p,q}(\mathbb{R}^d;E)}.$$

See [22, Section 3.2.2] (where the scalar case is considered) and [2].

In Theorem 3.2 below we shall obtain a version of Theorem 1.1 for bounded domains. We need the following lemma, where for $r > 0$ we denote $B_r := \{x \in E : \|x\| < r\}$.

Lemma 3.1. *Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$. There exists a constant C such that for every $r \geqslant 1$ and for all $f \in B^s_{p,q}(\mathbb{R}^d; E)$ with* $\operatorname{supp}(f) \subseteq B_r$,

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;E)} \leqslant C\|f|_{B_{2r}}\|_{B^s_{p,q}(B_{2r};E)}.$$

Proof. Choose $\psi \in \mathscr{S}(\mathbb{R}^d)$ such that $\psi \equiv 1$ on B_1 and $\psi \equiv 0$ outside B_2. Fix an integer $k > \max\{s, \frac{d}{p} - s\}$. Notice that for the $\frac{1}{r}$-dilation $\psi_{\frac{1}{r}}(x) := \psi(\frac{1}{r}x)$ we have $\|\psi_{\frac{1}{r}}\|_{W^{k,\infty}(\mathbb{R}^d)} \leqslant \|\psi\|_{W^{k,\infty}(\mathbb{R}^d)}$. Choose $g \in B^s_{p,q}(\mathbb{R}^d; E)$ such that $g \equiv f$ on B_{2r} and

$$\|g\|_{B^s_{p,q}(\mathbb{R}^d;E)} \leqslant 2\|f|_{B_{2r}}\|_{B^s_{p,q}(B_{2r};E)}.$$

Then it follows from the vector-valued generalization of [22, Theorem 2.8.2] that

$$\begin{aligned}\|f\|_{B^s_{p,q}(\mathbb{R}^d;E)} &= \|\psi_{\frac{1}{r}} f\|_{B^s_{p,q}(\mathbb{R}^d;E)} = \|\psi_{\frac{1}{r}} g\|_{B^s_{p,q}(\mathbb{R}^d;E)} \\ &\leqslant c\|\psi\|_{W^{k,\infty}(\mathbb{R}^d)}\|g\|_{B^s_{p,q}(\mathbb{R}^d;E)} \leqslant C\|f|_{B_{2r}}\|_{B^s_{p,q}(B_{2r};E)},\end{aligned}$$

where $C = 2c\|\psi\|_{W^{k,\infty}(\mathbb{R}^d)}$. □

Theorem 3.2. *Let $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$ and let $D \subseteq \mathbb{R}^d$ be a nonempty bounded open domain.*

(1) *E has type p if and only if we have a continuous embedding*

$$B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(D; E) \hookrightarrow \gamma(L^2(D), E).$$

(2) *E has cotype q if and only if we have a continuous embedding*

$$\gamma(L^2(D), E) \hookrightarrow B^{(\frac{1}{q}-\frac{1}{2})d}_{q,q}(D; E).$$

In both cases, the norm of the embedding does not exceed the norm of the corresponding embedding with D replaced by $\mathbb{R}^d$.

Note again the special cases corresponding to $p = 1$ and $q = \infty$, which hold for arbitrary Banach spaces E. Corollary 2.3 admits a version for bounded domains as well.

Proof. The "only if" parts in (1) and (2) and the final remark follow directly from the definition.

For the proofs of the "if" parts in (1) and (2), there is no loss of generality in assuming that $0 \in D$. Let $D_n = 2^n D$ and note that $\mathbf{1}_{D_n} \to \mathbf{1}$ pointwise. The idea is to 'dilate' the embedding for D to D_n and pass to the limit $n \to \infty$ to obtain the corresponding embedding for $\mathbb{R}^d$. That E has type p or cotype q is then a consequence of Theorem 1.1.

(1): The result being trivial for $p = 1$ we shall assume that $p \in (1, 2]$. Fix a function $f \in \mathscr{S}(\mathbb{R}^d; E)$ and note that by Lemma 2.1 (applied with $p = 1$) that $f \in \gamma(\mathbb{R}^d; E)$. Fix $n \geqslant 1$ arbitrary and put $f_n := f|_{D_n}$. Let $g_n : D \to E$ be defined by

$$g_n(x) := f_n(2^n x), \qquad x \in D.$$

Then $g_n \in \gamma(D; E)$ and

$$\|g_n\|_{\gamma(D;E)} = 2^{-\frac{1}{2}nd}\|f_n\|_{\gamma(D_n;E)}.$$

Also, $g_n = g^{(n)}|_D$, where $g^{(n)}(x) = f(2^n x)$ for $x \in \mathbb{R}^d$. By Lemma 1.2 there exists a constant $C > 0$, independent of n, such that

$$\|g_n\|_{B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(D;E)} \leqslant \|g^{(n)}\|_{B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;E)} \leqslant C2^{-\frac{1}{2}nd}\|f\|_{B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;E)}.$$

Denoting by $I : B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(D;E) \hookrightarrow \gamma(D;E)$ the embedding, it follows that

$$\begin{aligned}\|f_n\|_{\gamma(D_n;E)} &= 2^{\frac{1}{2}nd}\|g_n\|_{\gamma(D;E)} \\ &\leqslant 2^{\frac{1}{2}nd}\|I\|\,\|g_n\|_{B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(D;E)} \leqslant C\|I\|\,\|f\|_{B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;E)}.\end{aligned}$$

Passing to the limit $n \to \infty$ we obtain, by virtue of (1.2),

$$\|f\|_{\gamma(\mathbb{R}^d;E)} \leqslant C\|I\|\,\|f\|_{B^{(\frac{1}{p}-\frac{1}{2})d}_{p,p}(\mathbb{R}^d;E)}.$$

An application of Theorem 1.1 finishes the proof.

(2): It suffices to consider the case $q \in [2,\infty)$. Fix $f \in C_c^\infty(\mathbb{R}^d;E)$ and let $r \geqslant 1$ be so large that $\operatorname{supp}(f) \subseteq B_r$. With the same arguments as in (1) one can show that

$$\|f_n\|_{B^{(\frac{1}{q}-\frac{1}{2})d}_{q,q}(D_n;E)} \leqslant C\|f\|_{\gamma(\mathbb{R}^d;E)},$$

where $f_n = f|_{D_n}$ as before and C is a constant not depending on f and n. It follows from Lemma 3.1 that there is a constant C', independent of f and r, such that

$$\|f\|_{B^{(\frac{1}{q}-\frac{1}{2})d}_{q,q}(\mathbb{R}^d;E)} \leqslant C'\|f|_{B_{2r}}\|_{B^{(\frac{1}{q}-\frac{1}{2})d}_{q,q}(B_{2r};E)}.$$

Choosing n so large that $B_{2r} \subseteq D_n$, we may conclude that

$$\|f\|_{B^{(\frac{1}{q}-\frac{1}{2})d}_{q,q}(\mathbb{R}^d;E)} \leqslant C'\|f_n\|_{B^{(\frac{1}{q}-\frac{1}{2})d}_{q,q}(D_n;E)} \leqslant C'C\|f\|_{\gamma(\mathbb{R}^d;E)}.$$

Since $C_c^\infty(\mathbb{R}^d;E)$ is dense in $\gamma(\mathbb{R}^d;E)$ the result follows from Theorem 1.1. □

It is an interesting fact that at least in dimension $d = 1$, the "if part" of Theorem 3.2 (1) can be improved as follows.

Theorem 3.3. *If $p \in [1,2)$ is such that we have a continuous embedding*

$$B^{\frac{1}{p}-\frac{1}{2}}_{p,1}((0,1);E) \hookrightarrow \gamma(L^2(0,1),E),$$

then E has type p.

Proof. We may assume that $p \in (1,2)$.

First, for $s > 0$ we introduce an equivalent norm on $B^s_{p,q}(\mathbb{R};E)$ which does not involve the Fourier transform and can be handled quite easily from the computational point of view.

For $h \in \mathbb{R}$ and a function $f : \mathbb{R} \to E$ we define the function $T(h)f : \mathbb{R} \to E$ as the translate of f over h, i.e.

$$(T(h)f)(t) := f(t+h).$$

For $f \in L^p(\mathbb{R};E)$ and $t > 0$ let

$$\varrho_p(f,t) := \sup_{|h|\leqslant t} \|T(h)f - f\|_{L^p(\mathbb{R};E)}.$$

Then

$$\|f\|^*_{B^s_{p,q}(\mathbb{R};E)} := \|f\|_{L^p(\mathbb{R};E)} + \Big(\int_0^1 \big(t^{-s}\varrho_p(f,t)\big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}$$

(with the obvious modification for $q=\infty$) defines an equivalent norm on $B^s_{p,q}(\mathbb{R};E)$ (see [18, Proposition 3.1] or [20, Theorem 4.3.3]).

With these preliminaries out of the way we turn to the proof of the theorem. Since every Banach space has type 1 we may assume that $p\in(1,2)$. Let $n\geqslant 1$ and $x_0,\dots,x_{n-1}\in E$ be arbitrary and fixed. For $j=0,\dots,2n-1$, let $t_j=\frac{j}{2n}$. Define $f:\mathbb{R}\to E$ as

$$f=\sum_{k=0}^{n-1}\mathbf{1}_{(t_{2k},t_{2k+1}]}x_k.$$

Then $\|f\|_{L^p(\mathbb{R};E)}=(2n)^{-\frac{1}{p}}\big(\sum_{k=0}^{n-1}\|x_k\|^p\big)^{\frac{1}{p}}$. Let $0<t<(2n)^{-1}$ and take $0<|h|\leqslant t$. If $h>0$, then

$$T(h)f-f=\sum_{k=0}^{n-1}(\mathbf{1}_{(t_{2k}-h,t_{2k}]}-\mathbf{1}_{(t_{2k+1}-h,t_{2k+1}]})x_k.$$

If $h<0$, then

$$T(h)f-f=\sum_{k=0}^{n-1}(-\mathbf{1}_{(t_{2k},t_{2k}+h]}+\mathbf{1}_{(t_{2k+1},t_{2k+1}+h]})x_k.$$

In both cases we find that

$$\|T(h)f-f\|^p_{L^p(\mathbb{R};E)}\leqslant 2|h|\sum_{k=0}^{n-1}\|x_k\|^p\leqslant 2t\sum_{k=0}^{n-1}\|x_k\|^p.$$

This shows that $\varrho_p(f,t)\leqslant 2^{\frac{1}{p}}t^{\frac{1}{p}}\big(\sum_{k=0}^{n-1}\|x_k\|^p\big)^{\frac{1}{p}}$ for all $0<t<(2n)^{-1}$. It follows that

$$\begin{aligned}\int_0^{(2n)^{-1}}t^{-\frac{1}{p}+\frac{1}{2}}\varrho_p(f,t)\frac{dt}{t}&\leqslant 2^{\frac{1}{p}}\Big(\sum_{k=0}^{n-1}\|x_k\|^p\Big)^{\frac{1}{p}}\int_0^{(2n)^{-1}}t^{\frac{1}{2}}\frac{dt}{t}\\&=2^{\frac{1}{p}+1}(2n)^{-\frac{1}{2}}\Big(\sum_{k=0}^{n-1}\|x_k\|^p\Big)^{\frac{1}{p}}.\end{aligned}$$

If $t>(2n)^{-1}$, then $\varrho_p(f,t)\leqslant 2\|f\|_p=2(2n)^{-\frac{1}{p}}\big(\sum_{k=0}^{n-1}\|x_k\|^p\big)^{\frac{1}{p}}$. It follows that

$$\begin{aligned}\int_{(2n)^{-1}}^1 t^{-\frac{1}{p}+\frac{1}{2}}\varrho_p(f,t)\frac{dt}{t}&\leqslant 2(2n)^{-\frac{1}{p}}\Big(\sum_{k=0}^{n-1}\|x_k\|^p\Big)^{\frac{1}{p}}\int_{(2n)^{-1}}^1 t^{-\frac{1}{p}+\frac{1}{2}}\frac{dt}{t}\\&=2(2n)^{-\frac{1}{p}}\Big(\sum_{k=0}^{n-1}\|x_k\|^p\Big)^{\frac{1}{p}}\frac{1}{\frac{1}{p}-\frac{1}{2}}((2n)^{\frac{1}{p}-\frac{1}{2}}-1)\\&\leqslant 2(2n)^{-\frac{1}{2}}\frac{1}{\frac{1}{p}-\frac{1}{2}}\Big(\sum_{k=0}^{n-1}\|x_k\|^p\Big)^{\frac{1}{p}}.\end{aligned}$$

It follows that $f\in B^{\frac{1}{p}-\frac{1}{2}}_{p,1}(\mathbb{R};E)$ and by restricting to $(0,1)$ we obtain

$$\|f\|_{B^{\frac{1}{p}-\frac{1}{2}}_{p,1}((0,1);E)}\leqslant\|f\|_{B^{\frac{1}{p}-\frac{1}{2}}_{p,1}(\mathbb{R};E)}\leqslant C_p(2n)^{-\frac{1}{2}}\Big(\sum_{k=0}^{n-1}\|x_k\|^p\Big)^{\frac{1}{p}},$$

where C_p depends only on p. On the other hand,

$$\|I_f\|_{\gamma(L^2(0,1),E)} = (2n)^{-\frac{1}{2}} \Big\| \sum_{k=0}^{n-1} \gamma_k x_k \Big\|_{L^2(\Omega;E)}.$$

From the boundedness of the embedding $I : B^{\frac{1}{p}-\frac{1}{2}}_{p,1}((0,1);E) \hookrightarrow \gamma(L^2(0,1),E)$ we conclude that

$$(2n)^{-\frac{1}{2}} \Big\| \sum_{k=0}^{n-1} \gamma_k x_k \Big\|_{L^2(\Omega;E)} \leqslant C_p (2n)^{-\frac{1}{2}} \|I\| \Big(\sum_{k=0}^{n-1} \|x_k\|^p \Big)^{\frac{1}{p}}.$$

Hence E has type p, with Gaussian type p constant of at most $C_p\|I\|$. □

Returning to Theorem 3.2, we note the following consequence:

Corollary 3.4. *Let $D \subseteq \mathbb{R}^d$ be a nonempty bounded open domain with smooth boundary. Let $p \in [1,2]$ and $\alpha, \beta \in \mathbb{R}$ satisfy $\alpha > \beta + (\frac{1}{p} - \frac{1}{2})d \geqslant 0$. If E has type p, we have a continuous embedding*

$$C^\alpha(\overline{D};E) \hookrightarrow \gamma(H^{\beta,2}(D),E).$$

Proof. For $\alpha > \gamma > \beta + (\frac{1}{p} - \frac{1}{2})d \geqslant 0$ we have, cf. [2],

$$C^\alpha(\overline{D};E) \hookrightarrow B^\gamma_{\infty,\infty}(D;E) \hookrightarrow B^{\beta+(\frac{1}{p}-\frac{1}{2})d}_{p,p}(D;E).$$

The result now follows from Theorem 3.2. □

For dimension $d = 1$ we have the following converse:

Theorem 3.5. *Let E be a Banach space, and let $p \in (1,2)$ and $\alpha \in (0, \frac{1}{p} - \frac{1}{2})$. If $C^\alpha([0,1];E) \hookrightarrow \gamma(L^2(0,1);E)$, then E has type p.*

In particular this shows that in the spaces $E = l^p$ and $E = L^p(0,1)$, with $p \in [1,2)$, for every $\alpha \in (0, \frac{1}{p} - \frac{1}{2})$ there exist α-Hölder continuous functions which do not belong to $\gamma(L^2(0,1),E)$. Indeed, for such α we can find $p < p' < 2$ such that $\alpha \in (0, \frac{1}{p'} - \frac{1}{2})$, but both l^p and $L^p(0,1)$ fail type p'. A similar result holds for $E = c_0$ and $E = C([0,1])$ and $\alpha \in (0, \frac{1}{2})$. This improves the examples in [19], where only measurable functions are considered.

Proof. Assume for a contradiction that E is not of type p. We will show that this leads to a contradiction. By the Maurey-Pisier theorem (see [13]), l^p is finitely representable in E. Fix an integer n and let $T : l^p_n \to E$ be such that for all $x \in l^p_n$

$$\|x\|_{l^p_n} \leqslant \|Tx\| \leqslant 2\|x\|_{l^p_n}.$$

Choose $1 < r < (\frac{1}{2}p + \alpha p)^{-1}$. Let $c = \sum_{i \geqslant 1} i^{-r}$ and let $t_0 = 0$, $t_k = c^{-1} \sum_{i=1}^k i^{-r}$ for $k \geqslant 1$. Let $(e_k)_{k=1}^n$ be the standard basis of l^p_n and define $g_n : [0,1] \to l^p_n$ as

$$g_n(t) = \begin{cases} \Big(1 - \frac{|2t - t_k - t_{k-1}|}{t_k - t_{k-1}}\Big) e_k, & \text{if } t \in (t_{k-1}, t_k] \text{ for } 1 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

We claim that g_n is Hölder continuous of exponent α and

$$\begin{aligned} \|g_n\|_{C^\alpha([0,1];l^p_n)} &= \sup_{t \in [0,1]} \|g_n(t)\|_{l^p_n} + \sup_{0 \leqslant s < t \leqslant 1} \frac{\|g(t) - g(s)\|_{l^p_n}}{|t-s|^\alpha} \\ &\leqslant 1 + 4(t_n - t_{n-1})^{-\alpha} = 1 + 4c^\alpha n^{r\alpha}. \end{aligned}$$

To show this we consider several cases. First of all $\|g_n(t)\|_{l^p_n} \leqslant 1$ for all $t \in [0,1]$. If $t, s \in [t_{k-1}, t_k]$ for some $1 \leqslant k \leqslant n$,

$$\begin{aligned}\|g_n(t) - g_n(s)\|_{l^p_n} &= \Big|\frac{|2t - t_k - t_{k-1}|}{t_k - t_{k-1}} - \frac{|2s - t_k - t_{k-1}|}{t_k - t_{k-1}}\Big| \\ &\leqslant \frac{2|t-s|}{t_k - t_{k-1}} \leqslant \frac{2|t-s|^\alpha}{|t_k - t_{k-1}|^\alpha} \leqslant \frac{2|t-s|^\alpha}{|t_n - t_{n-1}|^\alpha}.\end{aligned}$$

If $s \in (t_{k-1}, t_k]$ and $t \in (t_k, t_{k+1}]$ for some $1 \leqslant k \leqslant n-1$, then by the above estimate and the concavity of $x \mapsto x^\alpha$,

$$\begin{aligned}\|g_n(t) - g_n(s)\|_{l^p_n} &\leqslant \|g_n(t) - g_n(t_k)\|_{l^p_n} + \|g_n(t_k) - g_n(s)\|_{l^p_n} \\ &\leqslant \frac{2|t-t_k|^\alpha}{|t_n - t_{n-1}|^\alpha} + \frac{2|t_k - s|^\alpha}{|t_n - t_{n-1}|^\alpha} \leqslant \frac{2^{2-\alpha}|t-s|^\alpha}{|t_n - t_{n-1}|^\alpha}.\end{aligned}$$

If $s \in (t_{l-1}, t_l]$ and $t \in (t_{k-1}, t_k]$ for $l+2 \leqslant k \leqslant n$ then

$$\|g_n(t) - g_n(s)\|_{l^p_n} \leqslant 2 \leqslant 2(t_{k-1} - t_l)^\alpha (t_n - t_{n-1})^{-\alpha} \leqslant 2(t-s)^\alpha (t_n - t_{n-1})^{-\alpha}.$$

For the other cases the estimate is obvious and we proved the claim. We have $g_n \in \gamma(L^2(0,1), l^p_n)$ and a standard square function estimate (cf. [16, Example 7.3]) gives

$$\begin{aligned}\|I_{g_n}\|^p_{\gamma(L^2(0,1),l^p_n)} &\geqslant K_p^p \sum_{k=1}^n \Big(\int_{t_{k-1}}^{t_k} \Big(1 - \frac{|2t - t_k - t_{k-1}|}{t_k - t_{k-1}}\Big)^2 dt\Big)^{\frac{p}{2}} \\ &= K_p^p \sum_{k=1}^n \Big(\frac{t_k - t_{k-1}}{2} \int_{-1}^1 (1-|s|)^2\, dt\Big)^{\frac{p}{2}} \\ &\geqslant 3^{-\frac{p}{2}} c^{-\frac{p}{2}} K_p^p \frac{2}{2-pr}((n+1)^{-\frac{pr}{2}+1} - 1),\end{aligned}$$

where K_p is a constant depending only on p. Define $f_n : [0,1] \to E$ as $f_n := Tg_n$. Then f_n is α-Hölder continuous and $I_{f_n} \in \gamma(L^2(0,1), E)$ with

$$\|f_n\|_{C^\alpha([0,1],E)} \leqslant 2\|g_n\|_{C^\alpha([0,1],l^p_n)} \leqslant 2(1 + 4c^\alpha n^{r\alpha}).$$

and

$$\|I_{f_n}\|^p_{\gamma(L^2(0,1),E)} \geqslant \|I_{g_n}\|^p_{\gamma(L^2(0,1),l^p_n)} \geqslant 3^{-\frac{p}{2}} c^{-\frac{p}{2}} K_p^p \frac{2}{2-pr}((n+1)^{-\frac{pr}{2}+1} - 1).$$

Since the inclusion operator $I : C^\alpha([0,1]; E) \to \gamma(L^2(0,1), E)$ is bounded we conclude that

$$3^{-\frac{1}{2}} c^{-\frac{1}{2}} K_p \frac{2^{\frac{1}{p}}}{(2-pr)^{\frac{1}{p}}}((n+1)^{-\frac{pr}{2}+1} - 1)^{\frac{1}{p}} \leqslant 2(1 + 4c^\alpha n^{r\alpha}).$$

Since we may take n arbitrary large, this implies $-\frac{r}{2} + \frac{1}{p} \leqslant r\alpha$, so $r \geqslant (\alpha p + \frac{p}{2})^{-1}$. But this contradicts the choice of r, and the proof is complete. □

After the completion of this paper, an improvement of Theorem 3.5 has been obtained in [14] where it is shown that if $p_0 \in [1,2)$ and $C^\alpha([0,1]; E) \hookrightarrow \gamma(L^2(0,1), E)$ for $\alpha = \frac{1}{p_0} - \frac{1}{2}$, then E has type p for some $p > p_0$.

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