1. Week 1: Introduction and conditional expectations

Study chapter 1 of the lecture notes. The results on functions of bounded variation will be discussed in more detail at a later moment, so 1.1.9 may be skipped at first reading. Also

Problem 1.1. Prove (i), (ii), (iii), (iv), (vi) of Theorem 1.27.

Problem 1.2. Let $\Omega = [0,1]$, $\mathscr{F} = \mathscr{B}([0,1])$ the Borel σ-field on $[0,1]$, and $\mathbb{P}$ the Lebesgue measure on $[0,1]$. Let $\mathscr{G}$ be the smallest σ-algebra on $[0,1]$ containing the Borel subsets of $[0,\frac{1}{2}]$. For $X \in L^1$ compute $\mathbb{E}(X|\mathscr{G})$.

Problem 1.3. Let $X : \Omega \to \mathbb{R}$ be a random variable in L^1.

(1) Prove that $|X|$ is $\sigma(X^2)$-measurable.
(2) Compute the conditional expectation $E(X \mid X^2)$ if:
   (i) X is almost surely non-negative.
   (ii) X is symmetric, i.e. X and $-X$ have the same distribution.
   Hint: Write $X = |X| \operatorname{sgn}(X)$, where $\operatorname{sgn}(X) = 1_{\{X>0\}} - 1_{\{X<0\}}$. Then show that for all Borel sets B in $\mathbb{R}$ one has

$$\int_{\{X^2 \in B\}} \mathbb{E}(1_{\{X>0\}}|X^2)\, d\mathbb{P} = \int_{\{X^2 \in B\}} \mathbb{E}(1_{\{X<0\}}|X^2)\, d\mathbb{P}.$$

Some properties of the conditional expectation which are more difficult to prove are the discussed in the next few problems:

Problem 1.4. Prove (ix) of Theorem 1.27. *Hint:* Use the fact that there exist $\alpha_n, \beta_n \in \mathbb{R}$ for $n \geq 1$ such that

$$f(x) = \sup_{n \geq 1} \alpha_n x + b_n, \quad \text{for } x \in (a,b).$$

Problem 1.5. Prove (x) of Theorem 1.27. You might first do exercise 1.14 from the lecture notes.

Problem 1.6. If $\mathcal{H}$ is independent of $\sigma(\sigma(X), \mathcal{G})$, then $\mathbb{E}(X|\sigma(\mathcal{G},\mathcal{H})) = \mathbb{E}(X|\mathcal{G})$.
Hint: Use Dynkin's $\pi - \lambda$ theorem from the appendix (Theorem B.3).

Good exercises conditional expectations from the lecture notes are:

- 1.13, 1.15, 1.16, 1.18.

Finally some measure theoretic exercises:

Problem 1.7. Let $X : \Omega \to \mathbb{R}$ be a random variable.

(1) Prove that if $Y : \Omega \to \mathbb{R}$ is $\sigma(X)$-measurable, then there exists a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Y = f(X)$.
   Hint: First consider the case where $Y = 1_A$ with $A \in \sigma(X)$.
(2) Is this function f unique?

Problem 1.8. Suppose the measurable functions f_n, f, g are defined on a measure space $(\Omega, \mathcal{F}, \mu)$ and satisfy $\lim_{n\to\infty} f_n = f$ pointwise μ-almost everywhere and $|f_n| \leq g$ for all n. Suppose further that $\int_\Omega |g|^p \, d\mu < \infty$ for some $p \in [1,\infty)$. Show that

$$\lim_{n\to\infty} \int_\Omega |f_n - f|^p \, d\mu = 0.$$

Hint: Use the dominated convergence theorem in a suitable way.

Some measure theoretic exercises from the lectures notes which I recommend:

- 1.7, 1.8, 1.9, 1.11, 1.14

2. Week 2

Problem 2.1. Prove that the sum of two stopping times is again a stopping time.

Problem 2.2. Prove that $\mathcal{F}_\tau$ as defined in (2.4) is a σ-algebra.

Exercises from the lectures notes:

- 1.1-1.2 and 2.1-2.10.