Exercises to “Applied Functional Analysis”

Exercises to Lecture 1

Here are some exercises about metric spaces. Some of the solutions can be found in my own additional lecture notes on Blackboard, as the exercises are just Lemmas in there. For the others one should always think of a proof in \( \mathbb{R}^d \) and then abstract to the general case. (The case of \( \mathbb{R}^d \) is extensively dealt with in LN, Chapter 2-4.) Although none of the exercises is deep or complicated, I have put a star * for the ones which seem a bit more difficult than the others.

**Exercise 1** Let \( d \) be the discrete metric on a space \( \Omega \).
Show that *every* subset of \( \Omega \) is both open and closed.
Show that every function \( f : \Omega \rightarrow \Omega' \) into another (general) metric space is continuous.
Show that each Cauchy sequence in \( \Omega \) converges.

In all the following exercises, \((\Omega, d)\) always denotes a general metric space.

**Exercise 2** Let \( U, V \) be open subsets of \( \Omega \). Show that \( U \cap V \) is also open.
Let \( F, G \) be closed subsets of \( \Omega \). Show that \( F \cap G \) is also closed.

**Exercise 3** Show that every convergent sequence is a Cauchy sequence.

**Exercise 4** Show that a Cauchy sequence is convergent if it has a convergent subsequence.

**Exercise 5** Let \((\Omega, d), (\Omega', d')\) be two metric spaces and \( f : \Omega \rightarrow \Omega' \) a mapping. Show that the following assertions are equivalent.

(i) \( f \) is continuous.

(ii) For every \( U \subset \Omega' \) open in \( \Omega' \) its inverse image \( f^{-1}(U) := \{ y \in \Omega \mid f(y) \in U \} \) is open in \( \Omega \).

**Exercise 6** Suppose that \((\Omega, d)\) is complete, and \( A \subset \Omega \) closed. Then \( A \) as a metric space in its own right (with the induced metric) is also complete.
Exercise 7 * Let $A \subset \Omega$. For $x \in \Omega$ the number

$$d(x, A) := \inf \{d(x, a) \mid a \in A\}$$

is called the **distance** of $x$ to $A$. Show that

$$|d(x, A) - d(y, A)| \leq d(x, y) \quad (x, y \in \Omega)$$

and hence that $d(\cdot, A) : \Omega \rightarrow \mathbb{R}$ is continuous.

Show that $A$ is closed if and only if the implication

$$d(x, A) = 0 \quad \Rightarrow \quad x \in A$$

holds for all $x \in \Omega$.

Exercise 8 * Show that if $(A_i)_i$ is an arbitrary family of closed subsets of $\Omega$ then the intersection

$$A := \bigcap_i A_i$$

is also closed. For an arbitrary subset $B \subset \Omega$, show that

$$\overline{B} = \bigcap \{A \mid B \subset A \text{ and } A \text{ is closed}\}.$$ 

Here are some other notions from metric space theory that sometimes are used, at least in the official lecture notes LN (see LN p.38).

Let $A \subset \Omega$. The point $x \in A$ is called an **interior point** of $A$ if there is $r > 0$ such that $B_r(x) \subset A$. Hence $A$ is open if $A$ consists entirely of interior points.

The point $x \in \Omega$ is called an **accumulation point** of $A$ if every $\varepsilon$-neighbourhood of $x$ contains a point of $A$, different from $x$. Equivalently, there is a sequence $(x_n)_n \subset A$ with $x_n \to x$, and $x_n \neq x$ for all $n \in \mathbb{N}$. (Can you prove this?)

The official lecture notes (LN p.38) define the closure of $A$ as the set consisting of $A$ united with all the accumulation points of $A$. Make sure that you understand that this is the same as the definition I gave in the lecture.
Exercises 2

Applied Functional Analysis

Exercises to Lecture 2

Exercise 1 Do all the exercises of Chapter 5 of the official lecture notes.

Let me comment on this:
If you do not find Exercise 5.3.1 trivial, then it is worth while doing it.
The most important exercise is 5.3.6 (completeness of \( \ell^\infty \)). Do it in any case!
Exercise 5.3.3 is easy, but not very important. Note that I have not introduced the \( p \)-norms on \( \mathbb{R}^d \), but you may view \( \mathbb{R}^d \) as a subspace of \( \ell^p \) (first \( d \) coordinates, rest equal to 0).
Exercise 5.3.4 is pure luxury. Do it only if you have done all the rest.
Exercise 5.4.5: do the first part. I personally do not regard the equality issue in Hölder’s inequality as very important.
Here is a hint for Exercise 5.3.7 a): try to find an estimate of the form \( \|x\|_2 \leq c \|x\|_1 \) for sequences \( x \) of real numbers. A similar remark is appropriate for part c) of this exercise.
Exercise 5.3.8 is very easy when you understand what to prove. It then requires only our characterization of \( \overline{A} \) of Lecture 1, and Proposition 4 of the current lecture.

The following exercises are also worth looking at. They consist mainly in providing information for which I had no time in the lecture.

Exercise 2 Let \((E, \| \cdot \|)\) be a normed space. Suppose that \( x_n \to x \) and \( y_n \to y \) in \( E \). Show that \( x_n + y_n \to x + y \).

Exercise 3 Let \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) be two norms on a vector space \( E \), with associated unit balls \( B_1, B_2 \), respectively. Show that the following assertions are equivalent for some \( m > 0 \):

- (i) \( \|x\|_1 \leq m \|x\|_2 \) for all \( x \in E \).
- (ii) \( B_2 \subset m B_1 \).

Exercise 4 Let \((x_n)_n \subset E\), where \( E \) is a normed space. Prove the following assertions:

- a) If \( x := \lim_{n \to \infty} \sum_{j=1}^n x_j \) exists, then \( \lim_{n \to \infty} x_n = 0 \).

- b) If \( E \) is a Banach space and \( \sum_{j=1}^\infty \|x_j\| < \infty \) then \( \lim_{n \to \infty} \sum_{j=1}^n x_j \) exists.

Exercise 5 Prove the two “calculus lemmas” used in the lecture to prove Minkowski’s and Hölder’s inequality.
Exercises to Lecture 3

Lecture 3 as I gave it covers the whole of LN Chapter 6 and Appendix B. You can therefore solve each of the exercises in LN 6.4.

Comment to LN 6.4.2: I think you may give two different elements $g, h$. Please note also that “motivate your answer” is nonsense. You can be motivated to do something, but an answer cannot. (Motivation is a state of mind.) What the text means is: justify your answer, give a reason for it, argue for it.

Here is a very simple exercise that is (in a less explicit form) in my lecture.

**Exercise 1** Let $e_n$ denote the sequence $e_n = (0, 0, \ldots, 0, 1, 0 \ldots)$, where the 1 is at position $n$. In function notation

$$e_n(m) = 1 \quad \text{if} \quad n = m, \quad e_n(m) = 0 \quad \text{if} \quad n \neq m.$$

Show that $e_n \in \ell^p$ and $\|e_n\|_p = 1$. Show also that $e_n \to 0$ pointwise.

And last, but nor least, please study the proof of the completeness of $B(I)$ given in LN 6.3.3. It is even simpler than what I did in the lecture, and you are supposed to fabric such a proof by yourself!

Exercises to Lecture 4

First of all: do not be concerned about Chapter 7 of LN. It will come, I promise!

Second: I plan to include a different proof of the Weierstrass theorem (LN 8.1.5), but I need some time for this.

Third: As most of the lecture is not official course material, you should definitely concentrate on solving the Assignment 1. Besides this you may do the Exercises LN 8.3.1, part a), and LN 8.3.3. The other exercises of LN Chapter 8 are postponed.

Fourth: Do not bother about how Borel sets look like. They can be weird, but I bet that practically every set you can possibly think of, is a Borel set. (Not that I have a better brain, but although one can prove that there are many many, non-Borel sets are more or less impossible to describe explicitly. There is a branch of maths, “descriptive set theory” which deals with such questions. As I said, do not bother too much.)

If you are keen to practice a bit the Lebesgue integration theory stuff, here are some (elementary) exercises.
Exercise 1  Given that all open subset of \( \mathbb{R} \) are Borel sets, show that all closed subsets and all kinds of intervals are also Borel sets.

Exercise 2  Show that the finite additivity in the definition of Lebesgue measure is superfluous because it follows from the \( \sigma \)-additivity and the requirement that \( \lambda(\emptyset) = 0 \).

Exercise 3  Show that a function \( f : I \rightarrow [-\infty, \infty] \) is measurable if and only if for each \( a \in \mathbb{R} \) the set
\[
\{ f \geq a \} = \{ x \in I \mid f(x) \geq a \}
\]
is a Borel set. (This gives an even easier criterion for measurability.)

Exercise 4  Using the previous exercise, show that if \( f, g \) are measurable, then also \( \max\{f, g\} \) is measurable.

Exercise 5  Show that if \( f : I \rightarrow [-\infty, \infty] \) is measurable, then the sets \( \{ f = \infty \} \) and \( \{ f = -\infty \} \) are Borel sets.

Exercise 6  Show that the Lebesgue measure is \textit{monotone}, i.e.
\[
A, B \in \mathfrak{B}(\mathbb{R}), \ A \subset B \quad \Rightarrow \quad \lambda(A) \leq \lambda(B).
\]
(Hint: write \( B = A \cup (B \setminus A) \) and use finite additivity.)

This last exercise is a bit more involved.

Exercise 7  Suppose \( (B_n)_n \) is a sequence of Borel subsets of \( \mathbb{R} \) such that \( B_1 \subset B_2 \subset \ldots \). As \( \mathfrak{B}(\mathbb{R}) \) is a \( \sigma \)-algebra, \( B := \bigcup_n B_n \) is again a Borel set. Show that
\[
\lambda(B) = \lim_n \lambda(B_n) = \sup_n \lambda(B_n)
\]
(Hint: use \textit{disjointification}, i.e. consider the sets \( A_1 := B_1, \ A_2 := B_2 \setminus B_1, \ A_3 := B_3 \setminus B_2 \) and use \( \sigma \)-additivity.)
Exercises to Lecture 5

We now covered the whole Chapter 8 from LN, except Example LN 8.2.13, which I shall do shortly. So in principle, you have all necessary information to do the Exercises of LN 8.3. Exercise 8.3.9 is totally superfluous, so you might skip it.

Here are some proposals for additional work.

Exercise 1 Prove Hölder’s inequality in the case $p = 1, q = \infty$.

Exercise 2 Show that if $f \in L^\infty(a, b)$, then $|f(x)| \leq \|f\|_{L^\infty}$ almost everywhere.

Exercise 3 Show that on $C[a, b]$ the norm $\|\cdot\|_{L^\infty}$ coming from $L^\infty(a, b)$ is the same as the uniform norm. Show that this is not true for other bounded functions.

Exercise 4 Prove that $L^\infty(a, b)$ is a Banach space.

Exercises to Lecture 6

In the current lecture, I covered almost the whole of LN Chapter 7. What I did not yet tell you is that two elements $x, y$ of an inner product space are called orthogonal if $\langle x, y \rangle = 0$. You need this information to be able to solve the exercises. In principle, you are now prepared to do all of them. However, leave out LN 7.4.6.

Try also to read the section LN 7.3 about the “control problem”. (I shall not treat this in my lectures for time reasons.) The following (simple) exercise is the abstract principle behind LN 7.3.

Exercise 1 Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space, and let $0 \neq f \in H$. Show that
\[
\|f\| = \sup\{\langle f, g \rangle \mid g \in H, \|g\| = 1\}
\]
and the supremum is attained at $g = f/\|f\|$.

Here is an exercise for the very keen among you.

Exercise 2 Let $H = L^2(-1, 1)$. Show that $f := 1_{(0,1)}$ does not have a weak derivative in $H$. (Note: Considered within the space $L^2(0, 1)$ it does have a weak derivative. Which?)
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Exercises to Lecture 7

(I apologize for the hurry, but it is really not my fault that the TUs decided to put a four-hour material into a two-hour course.)

We now have covered the material of LN Chapter 7 and 9 completely. When you think, Lemma 2 of Lecture 7 is not in LN, you are mistaken: it is LN 11.2.3. Similarly, Lemma 6 is just a pre-version of something which comes again in LN Chapter 11.

I admit that LN does not address ONS’s over arbitrary index sets, but for this reason it has to cheat you several times. Since I do not like cheating students, I decided to tell you the true story. But if you find it too difficult then just imagine all index sets to be countable or finite. However, you should be aware of the problem that a generic ONS, even if countable, does not come with a canonical enumeration. This is already clear with the trigonometric system, which gives you for each $n \in \mathbb{N}$ two functions, namely $\cos nx$ and $\sin nx$. The official LN do not say anything about that.

The issue of separable spaces is a further complication. The official lecture notes just say (on LN p.80): “every separable space has a maximal ONS”. But it was never defined what a separable space is.

The Gram-Schmidt procedure as detailed in LN and in my notes should already be known from undergraduate studies. Please look at Example LN 9.2.3 and do Exercises LN 9.4.1 and LN 9.4.2 to familiarize with it. From my point of view, it is just the application of orthogonal projections, and you definitely have to know this!

I find it very sad that there is no time to talk more in detail over concrete Fourier series. Certainly you already learned something about them in your undergraduate studies. One can use the theory to establish certain series representations as

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

see Example LN 9.3.7 and Exercise 9.4.3.

I find the Exercise LN 9.4.5 the most interesting one. It helps us to see that the trigonometric system is complete (see Theorem 18 in Lecture 7).

Here are some more exercises that you could do.

**Exercise 1** Prove that if $\langle e_i \rangle_{i \in I}$ is a maximal ONS in a Hilbert space $H$ then

$$\langle x, y \rangle = \sum_{i \in I} \langle x, e_i \rangle \cdot \langle e_i, y \rangle$$
for all \( x, y \in H \). (See LN 9.3.5)

**Exercise 2** Prove Lemma 9 of Lecture 7.

The last three are for the specialists.

**Exercise 3** Let \((e_n)_{n \in \mathbb{N}}\) be a complete ONS in the Hilbert space \(H\) and let \((f_i)_{i \in I}\) be a second ONS. Prove that \(I\) is also countable.

**Exercise 4** Use the fact that \(\mathbb{Q}\) is dense in \(\mathbb{R}\) to prove that a space \(E\) is separable if and only if \(E\) contains a countable dense set.

With these in the back you should be able, in one way or another, to do the last.

**Exercise 5** Let \(H\) be a separable Hilbert space and let \(F \subset H\) be a closed subspace. Show that \(F\) is separable.

**Exercises to Lecture 8**

As I said in the lecture: we postpone LN Chapter 10. Lecture 8 covered the whole of LN Chapter 11. So all exercises in LN 11.3 are suitable and worthwhile your attention.

**Exercise 1** Let \(E\) be a vector space, \(U, V\) be subspaces of \(E\) such that \(E = U + V\) and \(U \cap V = \{0\}\). I showed in the lecture that every \(x \in E\) can be written in a unique way as \(x = u + v\) with \(u \in U\) and \(v \in V\). Do you remember how this worked?

By this result, we may define mappings \(P_U : E \rightarrow U\) and \(P_V : E \rightarrow V\) such that \(x = P_U x + P_V x \) for all \(x \in E\). Show that \(P_U\) and \(P_V\) are linear and satisfy \(P_UP_U = P_U\), \(P_VP_V = P_V\).
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Exercises to Lecture 9

Now LN Chapter 10 is almost covered. What I did not do (yet) is: Integral operators (LN 10.1.6, 10.2.7), Hilbert-Schmidt operators (LN 10.1.7), and the notion of invertibility (LN 10.3.7, 10.3.8).

In principle all the exercises in LN 10.4 now accessible. In two cases you need the definition of invertibility from 10.3.4: a linear mapping \( T : E \rightarrow F \) is called invertible if it is bijective, i.e., one-one and onto. I included that definition in the additional lecture notes on page 76.

I unfortunately made a big mistake in the lecture when formulating the density principle (Lemma 8). Please look carefully at the corrected version: you cannot conclude the boundedness of \( T \) from the boundedness of \( T_0 \).

Here is an additional exercise to refresh your linear algebra.

**Exercise 1** Let \( E \) be a finite dimensional vector space with basis \{\( b_1, \ldots, b_d \)\}, and let \( T : E \rightarrow F \) be a linear mapping, where \( F \) is any vector space. Show that \( T \) is uniquely determined by the values \( T b_j \), \( j = 1, \ldots, d \). Conversely, show that to any vectors \( c_1, \ldots, c_d \in F \) there is a (unique) linear mapping \( T : E \rightarrow F \) such that \( T b_j = c_j \), \( j = 1, \ldots, d \).

The following two exercises show that in the definition of the operator norm

\[
\|T\| = \sup \{\|Tx\| \mid \|x\| \leq 1\}
\]

the supremum need not be a maximum! (The official lecture notes LN give a simpler example, see LN 10.2.6 d).)

**Exercise 2** Let \( E := \{f \in C[0,1] \mid f(1) = 0\} \), with supremum norm. This is a closed linear subspace of \( C[0,1] \) as it is the kernel of the bounded functional \( \delta_1 = (f \mapsto f(1)) \), see above. On \( E \) consider the multiplication operator \( T \) defined by \( (Tf)x = xf(x) \), \( x \in [0,1] \). Show that \( T \) is bounded with \( \|T\| = 1 \). Then show that \( \|Tf\|_\infty < \|f\|_\infty \) for every \( 0 \neq f \in E \).

**Exercise 3** Let again \( E := \{f \in C[0,1] \mid f(1) = 0\} \), with supremum norm. Consider the functional \( \varphi \) on \( E \) defined by integration:

\[
\varphi(f) := \int_0^1 f(x) \, dx \quad (f \in E).
\]

Show that \( \varphi \) is bounded with norm \( \|\varphi\| = 1 \). Then show that for every \( 0 \neq f \in E \) one has \( |\varphi(f)| < \|f\|_\infty \).
Exercises to Lecture 10

We covered LN Chapter 12 except for the part which deals with compact operators. We will come to this later. You may do Exercises LN 12.4.1 - 12.4.4 and 12.4.8. (The rest is over compact operators.) For LN 12.4.8 I give you a hint:

Exercise 1 Show that the set
\[ \{ A \in BL(E) \mid A \text{ is invertible} \} \]
is open in \( BL(E) \), when \( E \) is a Banach space.
(Hint: Let \( T \) be a bounded, invertible operator on \( E \) and let \( S \in BL(E) \) such that \( \|S - T\| < \|T^{-1}\| \). Write
\[ S = (\text{Id} - (T - S)T^{-1})T \]
and use the Neumann series to prove that \( S \) is invertible. What’s its inverse?)

Here are other proposals for exercises.

Exercise 2 The multiplication of operators

\[ [(S, T) \mapsto ST] : BL(F, G) \times BL(E, F) \longrightarrow BL(E, G) \]
is continuous: From \( \|T_n - T\| \to 0 \) and \( \|S_n - S\| \to 0 \) it follows that \( \|S_nT_n - ST\| \to 0 \). Prove this by imitate the proof of similar examples (scalar product on Hilbert spaces, scalar multiplication on a normed space) and using Lecture 10, Lemma 6.

Exercise 3 Here is the analogon on \( \ell^2 \) of an integral operator. Let \( a : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R} \) be an infinite matrix such that
\[ \|a\|_{HS} := \left( \sum_{i,j \in \mathbb{N}} |a(i, j)|^2 \right)^{1/2} < \infty \]
Show that \( a \) induces a linear operator \( A \) on \( \ell^2(\mathbb{N}) \) by
\[ (Af)(n) := \sum_{j=1}^{\infty} a(n, j)f(j) \]
and \( \|A\|_{\ell^2 \to \ell^2} \leq \|a\|_{HS} \).
Exercises to Lecture 11

I covered LN Chapter 13. However, I decided to replace LN 13.4 (the Galerkin method) by the treatment of Poisson’s equation on an interval. This is due to the fact that the material in LN 13.4.2 needs some information about $H^1(a,b)$ which is still missing. And as you know by now, I do not like to cheat you over the facts one is actually using.

Exercises LN 13.5 do not use the material of LN 13.4, and hence are all suitable to work on. Here are two additional ones.

**Exercise 1** Let $1 < p < \infty$ and let $q$ be its dual exponent. Show that $(\ell^p)^* = \ell^q$ by means of the natural duality

$$
\left\langle (f,g) \mapsto \sum_{n=1}^{\infty} f(n)g(n) \right\rangle : \ell^p \times \ell^q \rightarrow \mathbb{R}.
$$

(See Lecture 11, Theorem 3 or LN 13.3.1)

**Exercise 2** Let $\lambda > 0$ be fixed. Show that for every $f \in L^2(a,b)$ there is $u \in H^2(a,b)$ satisfying

$$
\lambda u - u'' = f, \quad u(a) = u(b) = 0.
$$

(Mimick the arguments from Lecture 11: define a new (suitable) scalar product on $H^1_0(a,b)$ and then use the Riesz-Fréchet theorem.) Suppose that we knew that $C^1_0[a,b]$ is dense in $H^1_0(a,b)$. Can we conclude that there is at most one solution of the boundary problem?
Exercises to Lecture 12

As I said in the lecture, I decided not to talk about complex Banach and Hilbert spaces. There are several reasons for that. First, I would not be able to do it in detail, and I do not like that since some things especially in the Hilbert space case change if you do everything for complex scalars. Not so much in the statements, but in the proofs. Then, I think that if you need complex spaces in the future, you will have no difficulty to acquire the needed knowledge on your own, as many standard books do it with complex scalars right from the start. Last, the complex world is not needed for our goal, which is the spectral theorem for compact self-adjoint operators on Hilbert spaces. And I prefer to present some substantial things in our remaining time, and not only technicalities.

Of course you are free to read a little in LN Chapter 14. And, of course, you may very well look at the exercises in LN 14.3. Especially LN 14.3.6 - 14.3.8 are also meaningful in the real setting, and are recommended.

I covered the notion of compact operator, a thing from LN 12.3 that I had skipped until now. My approach is a bit more general as in LN, but the most difficult thing (the diagonal argument in the proof of the closedness of $C(E, F)$) is also present in LN 12.3.5. Moreover, the definition of compactness given in LN is awkward, no serious book would define compactness like this.

If you know about the notion of compactness of a metric space, you may have noticed that there is connection with compact operators. (This is the reason for the name, of course.) But if you do not: never mind. It is not needed at all. Our definition of compact operator is all that we need.

Exercise LN 12.4.5 and LN 12.4.6 are helpful, I think.

Exercise 1 Complete the proof of Theorem 3, Lecture 12.

Exercise 2 Let $H$ be a Hilbert space, let $F \subset H$ be a closed linear subspace of $H$, and let $P = P_F$ be the orthogonal projection of $H$ onto $F$. Show that $P$ is compact if and only if $F$ is finite-dimensional.
Exercises to Lecture 13

I covered in principle the material of LN Chapter 15, but kept the general spectral theory much shorter. So it is worth-while to look at Examples LN 15.1.4, 15.1.8 and 15.2.3, 15.2.4, which are not included in my lecture notes. The Exercises of LN 15.4 except for 15.4.6 (which is a speciality for complex spaces) are all worth looking at. However, you will need the notion of a unitary linear mapping on a Hilbert space from LN 14.2.5.

I attached an appendix in which it is shown that the only spectral values of self-adjoint operators are approximate eigenvalues. The arguments are not difficult. There the following general fact is needed.

**Exercise 1** Let $E, F$ be Banach spaces and let $A : E \rightarrow F$ be a bounded linear mapping. Suppose that there exists $c \geq 0$ such that

$$
\|x\| \leq c\|Ax\| \quad (x \in E).
$$

Show that $\ker(A) = \{0\}$ and $\text{ran}(A)$ is closed.

Exercises to Lecture 14

In the last lecture I covered in principle the material from LN Chapter 16, but skipped the so-called Fredholm alternative (LN 16.2). LN Example 16.1.7 is in a complex-valued setting, so we could not treat this, but my examples on the Poisson problem with different boundary conditions are an ample replacement. LN Exercise 16.3.2 is included in the discussion of the norm of the integration operator.

If you want to read LN Chapter 17, you are free to do that, but it is not relevant for the exam.

Merry Christmas. And all the best for the exam and beyond!