Exercise 1

a) We have \( f_n := e_n - 2e_{n+2}, \ n \in \mathbb{N}, \) and \( S := \{ f_n \mid n \in \mathbb{N} \} \). To compute \( S^\perp \), take \( x \in \ell^2 \) arbitrary. Then \( x \perp S \) iff \( x \perp f_n \) for all \( n \in \mathbb{N} \) iff

\[
x_n - 2x_{n+2} = \langle x, e_n \rangle - 2 \langle x, e_{n+2} \rangle = \langle x, f_n \rangle = 0
\]

for all \( n \in \mathbb{N} \). This shows that \( x \in S^\perp \) iff

\[
x_{n+2} = \frac{1}{2} x_n \quad (n \in \mathbb{N})
\]

Hence the vectors

\[
h_1 := (1, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \ldots)
\]

\[
h_2 := (0, 1, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \ldots)
\]

are certainly in \( S^\perp \). On the other hand, given \( h \in S^\perp \) the vector

\[
y := x - x_1h_1 - x_2h_2
\]

satisfies \( y_1 = y_2 = 0 \) and \( y_{n+2} = \frac{1}{2}y_n \) for all \( n \geq 3 \). Hence \( y = 0 \) and so

\( x = x_1h_1 + x_2h_2 \). Consequently, \( S^\perp = \text{span}\{h_1, h_2\} \). But obviously \( h_1 \perp h_2 \), so \( \{h_1/\|h_1\|, h_2/\|h_2\|\} \) is an orthonormal system spanning \( F = S^\perp \).

b) For a general \( x \in \ell^2 \) we have

\[
P_Fx = \langle x, h_1/\|h_1\| \rangle \frac{h_1}{\|h_1\|} + \langle x, h_2/\|h_2\| \rangle \frac{h_2}{\|h_2\|} = \frac{\langle x, h_1 \rangle}{\|h_1\|^2} h_1 + \frac{\langle x, h_2 \rangle}{\|h_2\|^2} h_2.
\]

We need to compute \( \|h_1\|^2 \) and \( \|h_2\|^2 \). Since \( h_2 \) has the same “coordinates” as \( h_1 \) apart from an additional 0,

\[
\|h_1\|^2 = \|h_2\|^2 = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}
\]

Now specializing \( x = e_1 + e_2 \) we obtain \( \langle x, h_1 \rangle = 1 = \langle x, h_2 \rangle \) and hence

\[
P_Fx = \frac{1}{4/3} h_1 + \frac{1}{4/3} h_2 = \frac{3}{4} (h_1 + h_2) = \frac{3}{4} (1, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \ldots).
\]
Exercise 2 Let \( f \in L^p(a, b) \), and let \( q \) the conjugated exponent to \( p \), i.e. it holds \( \frac{1}{p} + \frac{1}{q} = 1 \). By Hölder’s inequality, for \( x \leq y \)
\[
|(Jf)(y) - (Jf)(x)| = \left| \int_x^y f(s) \, ds \right| \leq \int_x^y |f(s)| \, ds
\]
\[
\leq \left( \int_x^y \|f(s)|^p \, ds \right)^{1/p} \left( \int_x^y 1(s)^q \, ds \right)^{1/q}
\]
\[
\leq \|f\|_p |y - x|^{1/q} = \|f\|_p |y - x|^{1 - 1/p}.
\]
This shows that \( Jf \) is Hölder continuous with exponent \( \alpha = 1 - 1/p \), and
\[
\|f\|_{(\alpha)} \leq \|f\|_p \quad (f \in L^p(a, b)).
\]
Since \( (Jf)(a) = 0 \) is obvious, we proved \( f \in C^{1-1/p}_0[a, b] \) as claimed.

Exercise 3 a) If \( f = c \) is constant, then by the fundamental theorem of calculus
\[
\int_a^b cc' \, ds = c(\varphi(b) - \varphi(a)) = 0 = -\int_a^b 0 \cdot \varphi(s) \, ds
\]
for every \( \varphi \in C_0^1[a, b] \). Hence by definition, 0 is a weak derivative of \( f \). (Since we proved in the lectures that weak derivatives are unique, we may even say: 0 is the weak derivative of \( f \).)

Alternative: As \( f \) is constant, \( f \in C^1[a, b] \) with usual derivative \( f' = 0 \). In Lecture 6 (second remark after Definition 11) we proved that in this case the usual derivative is a weak derivative. Hence 0 is a (the) weak derivative of a constant function.

Remark: A common mistake here was to start with something like: “Let \( f \) be constant and \( g \) its weak derivative.” and in the end it is shown that \( g = 0 \). But: in general a weak derivative does not exist! And so one has only proved: IF a weak derivative of a constant function \( f \) exists, then it is zero. And this is not what was asked for.

b) Suppose that \( f \in C[a, b] \) can be written as \( f = \varphi' + c1 \) with \( c \in \mathbb{R} \) and \( \varphi \in C_0^1[a, b] \). Then
\[
\langle f, 1 \rangle = \langle \varphi', 1 \rangle + c \langle 1, 1 \rangle = c(b - a).
\]
This shows that \( c = (b-a)^{-1} \int_a^b f(s) \, ds \) is uniquely determined by \( f \). To prove the claim of the exercise, let \( f \in C[a, b] \) be arbitrary, define \( c := (b-a)^{-1} \langle f, 1 \rangle \) and \( g := f - c1 \). Then \( \varphi \) defined by
\[
\varphi(x) := \int_a^x g(s) \, ds \quad (x \in [a, b])
\]
clearly is in $C^1[a, b]$, $\varphi' = g$, and $\varphi(a) = 0$. But also
\[
\varphi(b) = \int_a^b g(s) \, ds = \langle g, 1 \rangle = \langle f - c_1, 1 \rangle = \langle f, 1 \rangle - c \langle 1, 1 \rangle = \langle f, 1 \rangle - \langle f, 1 \rangle = 0
\]
and hence $\varphi \in C^1_0[a, b]$.

**Remark:** A common mistake here was that there was no clear distinction between the first part of the answer (getting an idea how $\varphi$ and $c$ should be) and then proving the claim. Many of you actually proved: IF $f$ can be written in the form $f = \varphi' + c$ with $\varphi$, $c$ as required, THEN $c$ has the form . . . and $\varphi$ has the form . . . But strictly speaking, this is not what was asked for. And attention: if you use an implication sign as
\[
\Rightarrow \varphi = \ldots
\]
this has to be read as a statement about an object $\varphi$, that was already introduced. If you actually mean “Therefore, we can chose $\varphi := \ldots$ then you have to write that, and not use the form above!

c) We have seen in a) that $\varphi' \perp 1$ for every $\varphi \in C^1_0[a, b]$. This proves the inclusion $\supset$. To prove the reverse inclusion take $f \in H = L^2(a, b)$ such that $f \perp \varphi'$ for all $\varphi \in C^1_0[a, b]$. Define $e$ to be the constant function $e := (b-a)^{-1}1$. Thus $\langle 1, e \rangle = 1$. Now,
\[
f - \langle f, 1 \rangle e \perp \varphi'
\]
for all $\varphi \in C^1_0[a, b]$, but also
\[
f - \langle f, 1 \rangle e \perp 1.
\]
Hence by b),
\[
f - \langle f, 1 \rangle e \perp C[a, b].
\]
But $C[a, b]$ is dense in $L^2(a, b)$ and so
\[
f - \langle f, 1 \rangle e \perp L^2(a, b)
\]
since the scalar product is continuous (Lecture 6, Corollary 7 or LN 7.2.4). This implies that $f - \langle f, 1 \rangle e = 0$, i.e. $f = \langle f, 1 \rangle e = \langle f, 1 \rangle e = \langle f, 1 \rangle e = \frac{\langle f, 1 \rangle e}{b-a}1$ almost everywhere.

Here is a different way to prove the second half of c). Define $S := \{\varphi' \mid \varphi \in C^1_0[a, b]\}$ and again $e := (b-a)^{-1}1$, and fix $f \in S^\perp$. Since $C[a, b]$ is dense in $L^2(a, b)$, we find a sequence $(f_n)_n \subset C[a, b]$ such that $f_n \to f$ in the $L^2$-norm. Now, by 3b) we can write $f_n = \varphi_n' + c_n$, with certain $\varphi_n \in C^1_0[a, b]$ and $c_n \in \mathbb{R}$. Then
\[
\langle f_n, e \rangle = \langle \varphi_n', e \rangle + \langle c_n 1, e \rangle = 0 + c_n \langle 1, e \rangle = c_n.
\]
Define \( c := \langle f, e \rangle \), then \( \lim_n c_n = \lim_n \langle f_n, e \rangle = \langle f, e \rangle = c \). Consequently,

\[
\varphi'_n = f_n - c_n 1 \quad \text{in} \quad L^2(a, b).
\]

This shows that \( f - c 1 \in S \). But \( f, 1 \perp S \), and hence \( f, 1 \perp S \) (Lemma 2, Lecture 7). So \( f - c 1 \in S^\perp \cap S = \{0\} \). Hence \( f = c 1 \) is constant.

d) Suppose that \( f \in H^1(a, b) \) such that \( f' = 0 \). Then

\[
\langle f, \varphi' \rangle = -\langle f', \varphi \rangle = 0
\]

for all \( \varphi \in C^1_0[a, b] \), by the definition of a weak derivative. Now this is nothing else than

\[
f \perp \{ \varphi' \mid \varphi \in C^1_0[a, b] \}
\]

which by c) implies that \( f \) is a constant.

**Exercise 4**

a) Cauchy-Schwarz yields

\[
\|Jf\|^2_2 = \int_a^b \left| \int_a^t f(s) \, ds \right|^2 \, dt \leq \int_a^b \left( \int_a^t |f(s)|^2 \, ds \right) (t-a) \, dt
\]

\[
\leq \int_a^b (t-a) \, dt \, \|f\|^2_2 = \frac{(b-a)^2}{2} \|f\|^2_2.
\]

This yields \( \|Jf\|_2 \leq \frac{b-a}{\sqrt{2}} \|f\|_2 \) and hence that \( J : L^2(a, b) \rightarrow L^2(a, b) \) is continuous, (see Assignment 1, Exercise 1, or Lecture 9).

**Remark:** Other estimates are valid. For example

\[
\|Jf\|^2_2 = \int_a^b \left| \int_a^t f(s) \, ds \right|^2 \, dt \leq \int_a^b \left( \int_a^b |f(s)| \, ds \right)^2 \, dt
\]

\[
\leq \int_a^b \|f\|^2 (b-a) \, dt = (b-a)^2 \|f\|^2_2
\]

which leads to \( \|Jf\|_2 \leq (b-a) \|f\|_2 \). Also, one may use Exercise 2, which then amounts to similar estimates.

b) Let \( f \in C[a, b] \) and \( \varphi \in C^1_0[a, b] \). Then

\[
\langle Jf, \varphi' \rangle = \int_a^b \left( \int_a^t f(s) \, ds \right) \varphi'(t) \, dt
\]

\[
= \left( \int_a^b f(s) \, ds \right) \varphi(b) - \left( \int_a^a f(s) \, ds \right) \varphi(a) - \int_a^b f(s) \varphi(s) \, ds
\]

\[
= -\langle f, \varphi \rangle
\]
by ordinary integration by parts. Now let $f \in L^2(a,b)$ be arbitrary and fix $\varphi \in C^1_0[a,b]$. By the density of $C[a,b]$ in $L^2(a,b)$, there is a sequence $(f_n)_n \subset C[a,b]$ such that $\|f_n - f\|_2 \to 0$. Since $J$ is continuous with respect to the 2-norms, $\|Jf_n - Jf\|_2 \to 0$ as well. Since the scalar product is continuous (Lecture 6, Corollary 7 or LN 7.2.4),

$$\langle Jf_n, \varphi' \rangle \to \langle Jf, \varphi' \rangle \quad \text{and} \quad \langle f_n, \varphi \rangle \to \langle f, \varphi \rangle$$

as $n \to \infty$. By what we know already $\langle Jf_n, \varphi' \rangle = -\langle f_n, \varphi \rangle$. It follows that $\langle Jf, \varphi' \rangle = -\langle f, \varphi \rangle$

as claimed.

c) The statement in b) tells us that $f$ is a weak derivative of $Jf$, for any $f \in L^2(a,b)$. In particular, $Jf \in H^1(a,b)$. Since $1 \in H^1(a,b)$ anyway, we have the inclusion “$\supset$”. Now let $h \in H^1(a,b)$ be arbitrary. Define $f := h' \in L^2(a,b)$. Then $Jf$ and $h$ have the same weak derivative, i.e., the weak derivative of $h - Jf$ is 0. By Exercise 3 this implies that $h - Jf = c1$ for some $c \in \mathbb{R}$. This proves the other inclusion.

**Remark.** Sometimes the last argument was phrased in a somewhat different way: take $h \in H^1(a,b)$ and define $f = h'$ and $c := h(a)$. And then one tries to show that $h = Jf + c$. The mistake here is that the expression “$h(a)$” is not defined yet. Originally $h$ is just a function from $L^2(a,b)$ with some special property (it has a weak derivative), and as such is just an equivalence class of measurable functions. So you may change $h$ on the null-set $\{a\}$ as you like without effectively changing $h$ as an element of $L^2$.

The point is, that after this exercise, we will know that there is a continuous function $h_1$ such that $h = h_1$ almost everywhere. And then $h(a)$ is defined by using this (uniquely determined) continuous representative: $h(a) := h_1(a)$.

To sum up: The upshot of Exercises 3 and 4 is to justify the formula

$$h(t) = h(a) + \int_a^t h'(s) \, ds$$

for functions $h \in H^1(a,b)$, given that “$h(a)$” is meaningless in the first place.