THE FUNCTIONAL CALCULUS APPROACH TO OPERATOR COSINE FUNCTIONS

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Dedicated to N.K. Nikolski on the occasion of his retirement.

Abstract. In this article we study parabola-type operators and present the approach to (operator) cosine functions through the holomorphic functional calculus for such operators. This leads to a generation theorem for cosine functions in terms of the partial integrals in the complex inversion formula for the Laplace transform. Next, we develop a Hille-Phillips type functional calculus for cosine function generators and derive some new facts about the phase space group associated with the cosine function. Finally a transference principle is established that allows to derive nontrivial functional calculus estimates for cosine functions on Hilbert or, more general, on UMD spaces.

1. Introduction

A (operator) cosine function is a strongly continuous map $C : \mathbb{R} \to \mathcal{L}(X)$ of $\mathbb{R}$ into the bounded linear operators on a Banach space $X$ satisfying $C(0) = I$ and “d’Alembert’s equation”

$$C(t + s) + C(t - s) = 2C(t)C(s) \quad (s, t \in \mathbb{R}).$$

(1.1)

Cosine functions appear as solution families for well-posed second order abstract Cauchy problems

$$u'' + Au = 0 \quad u(0) = x, \ u'(0) = 0$$

like operator semigroups are the solution families for the first order Cauchy problem. The link between the cosine function $(C(s))_{s \in \mathbb{R}}$ and the operator $A$ is given by the Laplace transform

$$\lambda(\lambda^2 + A)^{-1} = \int_0^\infty e^{-\lambda s}C(s)\, ds$$

and $-A$ is then called the generator of the cosine function. If $-iB$ is the generator of a strongly continuous group $(U(s))_{s \in \mathbb{R}}$ then $C(s) := (U(s) + U(-s))/2$ is a cosine function whose negative generator is $A := B^2$. Not every cosine function appears in such a way, and an important result of Fattorini gives conditions for when it does. We refer to [1, Chap. 14] for all the background on cosine functions needed to understand the present paper and to [21] for a survey capturing the state of the art in 2004. Interesting recent results are presented in [18].

In semigroup theory, right from the start functional calculus methods played an important role, in the form of the so-called Hille-Phillips calculus. (For a general introduction to functional calculi we refer to [10].) This functional calculus is based
on integrals over the semigroup while it regards the generator as a derived concept. Alternatively, a holomorphic functional calculus is based on the Cauchy formula

\begin{equation}
    f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z) R(z, A) \, dz
\end{equation}

and uses nothing more than some growth condition on the resolvent of the generator. In the classical case of a bounded operator, this calculus is called the Dunford–Riesz calculus, and it was further developed for unbounded operators by Bade, McIntosh, deLaubenfels, the author and many others. However, although more general situations had been considered earlier by deLaubenfels [5], the holomorphic approach was used almost exclusively for sectorial operators (e.g. to treat holomorphic semigroups) and (sometimes) strip type operators (to treat \( \mathcal{C}_0 \)-groups). In an unpublished preprint [11] by the author, the notions of an operator of half-plane type and of parabola type were introduced and their relevance for semigroups and cosine functions were shown, cf. also [2]. The results about semigroups have in the meantime been incorporated in a recent paper [3] of Batty, Mubeen and the author. The results about cosine functions — considerably extended — appear here for the first time in printed form.

Here is a short synopsis of the paper. In Section 2 we briefly recall the notions of sectorial, half-plane and strip type operators. In Section 3 we introduce operators of weak parabola, parabola and strong parabola type. (The weak type operators did not appear in [11] and were included here to render more general the generation results Theorem 3.15 and Corollaries 3.16 and 3.17.) In Section 3.2 we present the construction of the holomorphic functional calculus for operators of (weak) parabola type, parallel to [12] and the existing constructions on strips, sectors and half-planes. In Section 3.3 we derive resolvent estimates for shifted operators of (strong) parabola type. Then, in Section 3.4, we show how to obtain an operator cosine function through the functional calculus. A corresponding generation theorem (Theorem 3.15) is established that allows to sharpen a recent result of Król [18, Thm. 2.6, (ii)⇒(i)], see Corollaries 3.16 and 3.17 and also Remarks 3.18.

In Section 4 we introduce the Hille-Phillips calculus for the negative generator of a cosine function and establish its compatibility with the holomorphic functional calculus. This is a generalization of the construction in [14] (called just ‘Phillips calculus’ there) in the case of bounded cosine functions. It parallels completely the construction of the Hille-Phillips calculus for strongly continuous groups in [12] and [15].

In Section 5, we establish new asymptotic properties of the sine function \( \sin(t) \) \( t \in \mathbb{R} \) associated with a cosine function \( \cos(t) \) \( t \in \mathbb{R} \) (Theorem 5.1 a) and b). These results allow to sharpen a result of [12] about the exponential growth of the associated group \( \mathcal{U} \) on the phase space. Finally, a transference principle for cosine functions is established. It allows to derive estimates for \( \| f(A) \| \), where \( -A \) is the generator of \( \cos(t) \) \( t \in \mathbb{R} \), in terms of Fourier multiplier norms on vector valued \( L^p \)-spaces. In case that the underlying space is UMD or Hilbert, such estimates are sharper than the generic estimates valid on every Banach space and lead to boundedness of certain functional calculi. (For more about this circle of ideas see [12, 14, 15] and in particular [17].)

1.1. Some Notations and Definitions. For a closed linear operator \( A \) on a complex Banach space \( X \) we denote by \( \text{dom}(A), \sigma(A) \) and \( \varrho(A) \) the domain, the spectrum and the resolvent set of \( A \), respectively. The space of bounded linear operators on \( X \) is denoted by \( \mathcal{L}(X) \). For two possibly unbounded linear operators \( A, B \) on \( X \) their product \( AB \) is defined on its natural domain \( \text{dom}(AB) := \{ x \in \text{dom}(B) \mid Bx \in \text{dom}(A) \} \). An inclusion \( A \subseteq B \) denotes inclusion of graphs, i.e., it means that \( B \)}
extends \( A \). A possibly unbounded operator \( A \) on \( X \) commutes with a bounded operator \( T \in \mathcal{L}(X) \) if \( \text{graph}(A) = T \times T \)-invariant, if and only if \( TA \subseteq AT \).

We let \( \mathbb{R}_+ := [0, \infty) \) and write \( \mathbb{C}_+ := \{ z \in \mathbb{C} \mid \text{Re} \, z > 0 \} \) for the open and \( \overline{\mathbb{C}_+} := \{ z \in \mathbb{C} \mid \text{Re} \, z \geq 0 \} \) for the closed right half-plane. For a domain \( \Omega \), let \( \mathcal{O}(\Omega) \) be the space of all holomorphic functions \( f : \Omega \to \mathbb{C} \), and \( H^\infty(\Omega) \) be the subspace of all bounded holomorphic functions. We denote the supremum norm on \( H^\infty(\Omega) \) by \( \| \cdot \|_{H^\infty(\Omega)} \) or \( \| \cdot \|_{H^\infty} \) if the context excludes ambiguities.

In the present paper we shall use standard results about vector-valued holomorphic functions as collected in [1, Appendix A].

2. Holomorphic Functional Calculi on Half-planes, Sectors, and Strips

This section has a preparatory character. We briefly review some well-known operator classes and their associated functional calculi, relevant for the treatment of general semigroups (operators of half-plane type), groups (strip type operators), fractional powers, logarithms and holomorphic semigroups (sectorial operators). These calculi have been treated extensively in [10] and [11, 3], and we refer to these sources for more information.

2.1. Operators of Half-Plane Type. Let us, for \( \omega \in [-\infty, \infty] \) denote by

\[
L_\omega := \{ z \in \mathbb{C} \mid \text{Re} \, z < \omega \}, \quad R_\omega := \{ z \in \mathbb{C} \mid \text{Re} \, z > \omega \},
\]

the open left and right half-planes defined by the abscissa \( \text{Re} \, z = \omega \), where in the extremal cases one half-plane is understood to be empty, and the other is the whole complex plane.

**Definition 2.1.** An operator \( A \) on a Banach space \( X \) is said to be of half-plane type \( \omega \in (-\infty, \infty] \) (in short: \( A \in \text{HP}(\omega) \)) if \( \sigma(A) \subseteq \overline{R_\omega} \) and

\[
M_\alpha := M_\omega(A) := \sup \{ \| R(z, A) \| \mid \text{Re} \, z \leq \alpha \} < \infty \quad \text{for every } \alpha < \omega.
\]

An operator \( A \) is of strong half-plane type \( \omega \) (in short: \( A \in \text{SHP}(\omega) \)) if

\[
M'_\alpha := M'_\omega(A) := \sup \{ |\alpha - \text{Re} \, z| \| R(z, A) \| \mid \text{Re} \, z \leq \alpha \} < \infty
\]

for every \( \alpha < \omega \). An operator \( A \) is said to be of (strong) half-plane type, if it is of (strong) half-plane type \( \omega \) for some \( \omega \in (-\infty, \infty] \).

Note that \( \text{SHP}(\omega) \subseteq \text{HP}(\omega) \). If \( A \) is of half-plane type, then it is of half-plane type \( s_0(A) \), where

\[
s_0(A) := \max \{ \omega \mid A \in \text{HP}(\omega) \} = \sup \{ \alpha \mid \sup_{\text{Re} \, z \leq \alpha} \| R(z, A) \| < \infty \}
\]

is the abscissa of uniform boundedness of the resolvent of \( A \).

For the construction of an elementary functional calculus for an operator of half-plane type, one employs (1.3) where \( f \) is a holomorphic function (of good decay) on a right half-plane and \( \Gamma \) is a vertical line, traversed top-down. This elementary calculus is then extended by regularization [10, 16] and one ends up with a meromorphic functional calculus in the sense of [10, Chap.1]. As a result, \( f(A) \) is defined for every \( f \in H^\infty(\mathbb{R}_\omega) \), \( \omega < s_0(A) \). In particular, one can define \( e^{-sA} := (e^{-s}I)(A) \) for \( s \geq 0 \). One then has the following basic generation result.

**Proposition 2.2.** Let \( A \) be an operator of half-plane type. Then \( -A \) is the generator of a strongly continuous semigroup if and only if \( \text{dom}(A) \) is dense, \( e^{-sA} \in \mathcal{L}(X) \) for all \( 0 \leq s \leq 1 \) and \( \sup_{0 \leq s \leq 1} \| e^{-sA} \| < \infty \).

We refer to [3] for details and more results concerning half-plane type operators and generation theorems.
2.2. Sectorial Operators. For $\omega \in [0, \pi]$ we let

$$S_\omega := \begin{cases} \{z \neq 0 \mid \arg z < \omega\}, & (\omega > 0) \\ (0, \infty), & (\omega = 0). \end{cases}$$

be the open sector of angle $2\omega$ symmetric about the positive real axis, with vertex at the origin. The following definition is classical.

**Definition 2.3.** An operator $A$ on a Banach space $X$ is called sectorial of angle $\omega < \pi$ (in short: $A \in \text{Sect}(\omega)$) if $\sigma(A) \subseteq \overline{S_\omega}$ and for every $\omega' \in (\omega, \pi]$ there is $M_{\omega'}$ such that

$$\|\lambda R(\lambda, A)\| \leq M_{\omega'} \quad (|\arg \lambda| \in [\omega', \pi]).$$

The least $\omega$ such that $A \in \text{Sect}(\omega)$ is denoted by $\omega_{\text{sec}}(A)$.

For sectorial operators there is a natural meromorphic functional calculus constructed in exactly the same way as for operators of half-plane type [10, Chap. 2]. Then an operator $-A$ generates a (sectorially) bounded holomorphic semigroup if and only if $A$ is sectorial of type $\omega_{\text{sec}}(A) < \pi/2$, see, e.g., [10, Sect. 3.4].

If an operator is sectorial and at the same time of half-plane type, and the operator $\lambda^s (A)$ is defined in either calculus, then it is the same in either definition. (This follows from setting up a new functional calculus on half-planes intersected with sectors and employing a basic lemma on “morphisms” of calculi [10, Sec.1.2.3].) In particular, this applies if $A$ satisfies an estimate of the form

$$\|R(\lambda, A)\| \lesssim \frac{1}{|\text{Re} \lambda|} \quad (\text{Re} \lambda < 0),$$

because in this case, $A$ is of strong half-plane type 0 and sectorial of angle $\pi/2$. Note that if $A$ is sectorial of angle $\omega_{\text{sec}}(A) < \pi/2$, then clearly (2.1) holds. Hence the operators $e^{-st}(A)$ for $s > 0$ can be defined in either calculus, and these definitions yield the same operators.

2.3. Strip Type Operators. For $\omega > 0$ we define the strip

$$\text{St}_\omega := \{z \in \mathbb{C} \mid |\text{Im} z| < \omega\}$$

and $\text{St}_0 := \mathbb{R}$.

**Definition 2.4.** An operator $A$ on a Banach space $X$ is said to be of strip type $\omega \geq 0$ (in short: $A \in \text{Strip}(\omega)$) if $\sigma(A) \subseteq \overline{\text{St}_\omega}$ and for any $\omega' > \omega$ there is $M_{\omega'} \geq 0$ such that

$$\|R(z, A)\| \leq M_{\omega'} \quad (|\text{Im} z| \geq \omega').$$

The least $\omega$ such that $A \in \text{Strip}(\omega)$ is denoted by $\omega_{\text{st}}(A)$.

An operator $A$ is of strong strip type $\omega \geq 0$ (in short: $A \in \text{SStrip}(\omega)$), if $\sigma(A) \subseteq \overline{\text{St}_\omega}$ and for every $\omega' > \omega$ there is $M_{\omega'}$ such that

$$\|R(\lambda, A)\| \leq \frac{M_{\omega'}}{|\text{Im} \lambda| - \omega'} \quad (|\text{Im} \lambda| > \omega').$$

The least $\omega$ such that $A \in \text{SStrip}(\omega)$ is denoted by $\omega_{\text{sst}}(A)$.

For example, if $-iA$ generates a group $(U(t))_{t \in \mathbb{R}}$ such that $\|U(t)\| \leq Me^{\omega|t|}$ then $A$ is of strong strip type $\omega$. One can weaken the hypothesis; in fact it suffices that $\theta(U) \leq \omega$, where

$$\theta(U) := \inf\{\omega \geq 0 \mid \exists M \geq 1 : \|U(t)\| \leq Me^{\omega|t|} \quad (t \in \mathbb{R})\}.$$ 

is the exponential group type.

We can associate a natural holomorphic functional calculus with a strip type operator $A \in \text{Strip}(\omega)$. One considers functions $f$ holomorphic on strips $\text{St}_\varphi$ with $\varphi > \omega$. If a function $f$ has integrable decay at $\pm \infty$, e.g., $f = O(|z|^{-1+s})$ as $|z| \to \infty$
for some \( s > 0 \), then \( f(A) \) is defined by the usual Cauchy integral (1.3), the contour \( \Gamma \) being the oriented boundary of an appropriate strip. This gives an elementary functional calculus for \( A \) which is extended to a meromorphic functional calculus by regularization, see [10, Sect. 4.2]. Of course, this functional calculus extends the two half-plane calculi available for \( \pm A \).

It is clear that a result analogous to Proposition 2.2 holds. The operator \(-iA\) generates a \( C_0\)-group if and only if \( A \) is densely defined, of strip type, and \((e^{itA})(A)\) is bounded for all \( t \in \mathbb{R} \).

The link between sectorial and strip type operators is as follows. It was essentially proved by Nollau that if \( A \) is an injective sectorial operator of angle \( \omega \) then \( \log(A) \) is of strong strip type \( \omega \). In fact, it was proved in [8] that the strip type of \( \log(A) \) is equal to the sectoriality type of \( A \): \( \omega_{st}(\log(A)) = \omega_{st}(\log(A)) = \omega_{sc}(A) \). Moreover, the spectral mapping theorem \( \sigma(\log(A)) = \log(\sigma(A) \setminus \{0\}) \) was established in [9]. One can switch back and forth from \( A \) to \( \log(A) \), as far as the functional calculi are concerned. Indeed, there is a composition rule

\[
 f(\log A) = (f(\log z))(A)
\]

in the sense that \( f(\log(A)) \) is defined (by the f.c. for strong strip type operators) if and only if \( f(\log z))(A) \) is defined (by the f.c. for sectorial operators). The operator \(-i\log(A)\) generates a \( C_0\)-group if and only if \( A \) has bounded imaginary powers \((A^s)_{s \in \mathbb{R}}\). (Since \( e^{is\log z} = z^s \) and in view of the composition rule above, this does not come as a surprise.) The fact that there are sectorial operators without bounded imaginary powers [10, Cor. 9.1.8] hence yields the existence of natural examples of strong strip type operators even on Hilbert spaces that do not generate a group.

The symmetry between (injective) sectorial and strip type operators, however, is only partial. There are strong strip type operators that fail to be logarithms of sectorial ones, e.g., the operator \(-d/dt\) on \( L^1(\mathbb{R}) \). Here, the best result up to now is due to Monniaux [19]: \(-iA\) generates a \( C_0\)-group \( U \) of group type \( \theta(U) < \pi \) and if the space \( X \) is UMD, then \( A \) is the logarithm of a sectorial operator. See [10, Chap. 4] and [12] for further information.

3. Parabola Type Operators and Cosine Functions

Now we turn to the treatment of operator cosine functions by (holomorphic) functional calculus. As a guiding intuition, we think of a generator \(-A\) of a cosine function as \(-A = (-iB)^2\) where \(-iB\) generates a group. It is then natural to consider a functional calculus on parabolas

\[
 \Pi_\omega := \{ z^2 \mid z \in St_\omega \} \quad (\omega \geq 0).
\]

To define an operator of parabola type \( \omega \) we need to specify a resolvent estimate that should hold outside every larger parabola. A natural way to find such a condition is to look at the negative generator of a cosine function.

3.1. Parabola Type Operators. So let \(-A\) generate a cosine function \((C(t))_{t \in \mathbb{R}}\) on the Banach space \( X \). General theory [1, Chapter 14] yields \( \omega \geq 0 \) and \( M \geq 1 \) such that \( \|C(t)\| \leq Me^{\omega t} \) for all \( t \in \mathbb{R} \). By (1.2)

\[
 \lambda R(\lambda^2, -A) = \int_0^\infty e^{-\lambda s}C(s)\,ds \quad (\text{Re} \lambda > \omega).
\]

Taking norms and estimating yield

\[
 \|R(-\lambda^2, A)\| \leq \frac{M}{|\lambda| (\text{Re} \lambda - \omega)} \quad (\text{Re} \lambda > \omega).
\]

\[
 (3.1)
\]
The function \((z \mapsto z^2) : \{ \text{Im} \, z > \omega \} \to \mathbb{C} \setminus \mathbb{R} \) is biholomorphic, its inverse being a branch of the square root. Writing \(\mu = -\lambda^2\) in (3.1) yields
\[
\|R(\mu, A)\| \leq \frac{M}{\sqrt{|\mu|} (|\text{Im} \sqrt{|\mu|} - \omega|)} \quad (\mu \notin \Pi_\omega).
\]
This expression is actually independent of the branch of the square root we take. It yields the canonical resolvent estimate for a (strong) parabola type operator.

**Definition 3.1.** An operator \(A\) on a Banach space \(X\) is said to be of weak parabola type \(\omega \geq 0\) (in short: \(A \in \text{WPara}(\omega)\)) if \(\sigma(A) \subseteq \Pi_\omega\) and for each \(\omega' > \omega\) there exists \(M_{\omega'}\) such that
\[
(3.2) \quad \|R(\mu, A)\| \leq M_{\omega'} \quad (\mu \notin \Pi_{\omega'}). \tag{3.2}
\]
The least \(\omega \geq 0\) such that \(A \in \text{WPara}(\omega)\) is denoted by \(\omega_{\text{wpa}}(A)\).

An operator \(A\) on a Banach space \(X\) is said to be of parabola type \(\omega \geq 0\) (in short: \(A \in \text{Para}(\omega)\)) if \(\sigma(A) \subseteq \Pi_\omega\) and for each \(\omega' > \omega\) there exists \(M_{\omega'}\) such that
\[
(3.3) \quad \|R(\mu, A)\| \leq \frac{M_{\omega'}}{\sqrt{|\mu|} (|\text{Im} \sqrt{|\mu|} - \omega'|)} \quad (\mu \notin \Pi_{\omega'}). \tag{3.3}
\]
The least \(\omega \geq 0\) such that \(A \in \text{Para}(\omega)\) is denoted by \(\omega_{\text{pa}}(A)\).

An operator \(A\) on a Banach space \(X\) is said to be of strong parabola type \(\omega \geq 0\) (in short: \(A \in \text{SPara}(\omega)\)) if \(\sigma(A) \subseteq \Pi_\omega\) and for each \(\omega' > \omega\) there exists \(M_{\omega'}\) such that
\[
(3.4) \quad \|R(\mu, A)\| \leq \frac{M_{\omega'}}{\sqrt{|\mu|} (|\text{Im} \sqrt{|\mu|} - \omega'|)} \quad (\mu \notin \Pi_{\omega'}). \tag{3.4}
\]
The least \(\omega \) such that \(A \in \text{SPara}(\omega)\) is denoted by \(\omega_{\text{spra}}(A)\).

It is clear that \(\text{SPara}(\omega) \subseteq \text{Para}(\omega) \subseteq \text{WPara}(\omega)\) and \(\omega_{\text{wpa}}(A) \leq \omega_{\text{pa}}(A) \leq \omega_{\text{spra}}(A)\).

We have seen above that if \(-A\) generates a cosine function of exponential growth type \(\omega\) then \(A\) is of strong parabola type \(\omega\). We now give two more classes of examples.

**Lemma 3.2.** Let \(\omega \geq 0\) and let \(B\) be an operator on \(X\) with \(\sigma(B) \subseteq \text{St}_\omega\).

a) If \(\|R(\lambda, B)\| \leq M\) for \(|\text{Im} \lambda| > \omega\) then
\[
\|R(\mu, B^2)\| \leq \frac{M}{\sqrt{|\mu|}} \quad (\mu \notin \Pi_\omega).
\]
b) If \(\|R(\lambda, B)\| \leq \frac{M}{|\text{Im} \lambda| - \omega}\) for \(|\text{Im} \lambda| > \omega\) then
\[
\|R(\mu, B^2)\| \leq \frac{M}{\sqrt{|\mu|} (|\text{Im} \sqrt{|\mu|} - \omega|)} \quad (\mu \notin \Pi_\omega).
\]

In particular: if \(B \in \text{Strip}(\omega)\), then \(B^2 \in \text{Para}(\omega)\); and if \(B \in \text{SStrip}(\omega)\), then \(B^2 \in \text{SPara}(\omega)\).

**Proof.** Let \(\mu \in \mathbb{C} \setminus \Pi_\omega\) and \(\lambda := \sqrt{|\mu|}\) (either choice). Then
\[
R(\mu, B^2) = (\lambda^2 - B^2)^{-1} = -R(\lambda, B)R(-\lambda, B) = \frac{1}{2\lambda} (R(\lambda, B) - R(-\lambda, B))
\]
by the resolvent identity. Estimating the norm and employing the hypotheses proves the claim.

We note the following geometric inequality.
Lemma 3.3. For $\omega \geq 0$ and $\mu \notin \Pi_{\omega}$ one has
\[
\frac{1}{\text{dist}(\mu, \Pi_{\omega})} \leq \frac{1}{\sqrt{|\mu|} \left( |\text{Im}\sqrt{|\mu|} - \omega| \right)}.
\]

Proof. We write $\mu = \lambda^2$, with $|\text{Im}\lambda| > \omega$. It suffices to show that $|\lambda| (|\text{Im}\lambda| - \omega) \leq \text{dist}(\lambda^2, \Pi_{\omega})$, i.e.
\[
|\lambda^2 - z^2| = |\lambda - z| |\lambda + z| \geq |\lambda| (|\text{Im}\lambda| - \omega)
\]
for all $z$ such that $|\text{Im}\lambda| \leq \omega$. But both the inequalities $|z - \lambda| \geq |\text{Im}\lambda| - \omega$ and $|z + \lambda| \geq |\text{Im}\lambda| - \omega$ hold, and at least one of the inequalities $|\lambda - z| \geq |\lambda|$ and $|\lambda + z| \geq |\lambda|$ holds, for trivial geometric reasons.

Corollary 3.4. Let $\omega > 0$. Then in the following cases one has $\sigma(A) \subseteq \Pi_{\omega}$ and
\[
\|R(\mu, A)\| \leq \frac{1}{\sqrt{|\mu|} \left( |\text{Im}\sqrt{|\mu|} - \omega| \right)} \quad (\mu \notin \Pi_{\omega})
\]

a) $A$ is an operator on a Hilbert space such that its numerical range and spectrum are contained in $\Pi_{\omega}$;
b) $A$ is a normal operator on a Hilbert space, with spectrum contained in $\Pi_{\omega}$;
c) $A$ is a multiplication operator on some $L^p$-space with a function $a$ with its essential range being contained in $\Pi_{\omega}$.

Clearly, this list could be prolonged, for instance by other examples of multiplication operators.

3.2. The Holomorphic Functional Calculus. We construct a functional calculus for a weak parabola type operator $A \in \text{WPara}(\omega)$ in the obvious fashion. Let $\varphi > \omega$ and let
\[
\mathcal{E}(\Pi_{\varphi}) := \left\{ f \in \mathcal{O}(\Pi_{\varphi}) \mid f(z) = O(|z|^{-\epsilon}) \text{ as } |z| \to \infty, \text{ for some } \epsilon > 1 \right\}.
\]

Then
\[
f(a) = \frac{1}{2\pi i} \int_{\partial \Pi_{\varphi}} f(z) \frac{dz}{z - a} \quad (a \in \Pi_{\omega'}),
\]
by Cauchy’s theorem, where $\omega' \in (\omega, \varphi)$. We define
\[
f(A) := \frac{1}{2\pi i} \int_{\partial \Pi_{\varphi}} f(z) R(z, A) dz = \frac{-1}{\pi i} \int_{\omega' + \infty}^{\omega' - \infty} f(z^2) z R(z^2, A) \frac{dz}{z}
\]
and as usual this does not depend on the choice of $\omega'$. In the usual fashion one shows that this defines an algebra homomorphism
\[
\Phi := (f \mapsto f(A)) : \mathcal{E}(\Pi_{\varphi}) \longrightarrow \mathcal{L}(X),
\]
and that it respects resolvents:
\[
[(\mu - z)^{-1} (\lambda - z)^{-1}](A) = R(\mu, A) R(\lambda, A) \quad (\lambda, \mu \notin \Pi_{\varphi}).
\]
Therefore, a meromorphic functional calculus in the sense of [10, Sect. 1.3] is defined, and there is a canonical definition of $f(A)$ for meromorphic functions $f$ on $\Pi_{\varphi}$ that are regularizable by elements of $\mathcal{E}(\Pi_{\varphi})$. Of course, one obtains a corresponding convergence lemma in the case that $A$ is densely defined, cf. [10, Sec.5.1] and [3, Thm.3.1].

Remark 3.5. Suppose that the weak parabola type operator $A$ is actually of parabola type $\omega$, let $\varphi > \omega$ and let $f \in \mathcal{O}(\Pi_{\varphi})$ such that
\[
f(z) = O(|z|^{-\epsilon}) \text{ as } |z| \to \infty, \text{ for some } \epsilon > 1/2.
\]
Then the integrals in (3.5) are still absolutely convergent, due to the stronger estimates (3.3) in place of just (3.2). Then \( f(A) \in \mathcal{L}(X) \) and \( f(A) \) is given by the integrals in (3.5). This is shown by using a function \((\mu - z)^{-1} (\mu \notin \Pi_\varphi)\) as a regularizer and employing the identity
\[
\frac{f(z)}{\mu - z} R(z, A) = R(\mu, A) \left( \frac{f(z)}{\mu} R(z, A) + \frac{f(z)}{\mu - z} \right)
\]
and Cauchy’s theorem. Hence, in this case one may as well use the class
\[
\mathcal{E}'(\Pi_\varphi) := \left\{ f \in \mathcal{O}(\Pi_\varphi) \mid f(z) = O(|z|^{-\epsilon}) \text{ as } |z| \to \infty, \text{ for some } \epsilon > 1/2 \right\}
\]
as the class of “elementary” functions for defining the functional calculus, cf. [10, Prop. 1.2.5].

If the parabola type operator \( A \) arises as a square \( A = B^2 \) of a strip type operator \( B \), then there is an obvious composition rule:

**Proposition 3.6.** Let \( B \in \text{Strip}(\omega) \) and \( A := B^2 \). Let \( \varphi > \omega \) and \( f \in \mathcal{M}(\Pi_\varphi) \) such that \( f(A) \) is defined. Then \( [f(z^2)](B) \) is defined and \( f(z^2)(B) = f(A) \).

**Proof.** By [10, Prop. 1.3.6] one may suppose without loss of generality that \( f \in \mathcal{E}(\Pi_\varphi) \). Then one may perform a computation similar to [10, p.43,p.96/97]. But it is even simpler:
\[
f(A) = \frac{1}{2\pi i} \int_{\partial \Pi_\varphi} f(z) R(z, A) \, dz = \frac{1}{2\pi i} \int_{\text{Im } z = \omega'} f(z^2) 2z R(z^2, B^2) \, dz
\]
\[
= \frac{1}{2\pi i} \int_{\text{Im } z = \omega'} f(z^2) 2z R(z^2, B^2) \, dz
\]
\[
= \frac{1}{2\pi i} \int_{\text{Im } z = \omega'} f(z^2) [R(z, B) - R(-z, B)] \, dz
\]
\[
= \frac{1}{2\pi i} \int_{\text{Im } z = \omega'} f(z^2) R(z, B) \, dz = [f(z^2)](B)
\]
with the appropriate orientations of the contours. \( \square \)

### 3.3. Shifting to the Right.
The classes of operators of half-plane, strip or sector type are invariant under shifting the operator to the right. We aim at an analogous result for parabola type operators. To facilitate estimates we shall need the following lemma.

**Lemma 3.7.** For \( \lambda \in \mathbb{C} \) one has
\[
|\text{Im } \sqrt{\lambda}| = \frac{|\lambda| - \text{Re } \lambda}{2}
\]
where \( \sqrt{\lambda} \) denotes any choice of square-root of \( \lambda \) and \( \varphi = \text{arg } \lambda \). Consequently,
\[
|\text{Im } \sqrt{\lambda}| = \frac{|\lambda| - \text{Re } \lambda}{2} = \frac{|\text{Im } \lambda|}{\sqrt{|\lambda| \sqrt{2} + 2 \cos \varphi}}
\]
if \( \lambda \notin (-\infty, 0] \). Moreover,
\[
\partial \Pi_\omega = \{ \lambda \mid |\lambda| = \text{Re } \lambda + 2\omega^2 \} \quad \text{and} \quad \Pi_\omega = \{ \lambda \mid |\lambda| \leq \text{Re } \lambda + 2\omega^2 \}
\]
for any \( \omega \geq 0 \).

**Proof.** Let \( \lambda = a + ib \) and \( \sqrt{\lambda} = x + iy \). Then \( x^2 - y^2 = a \) and \( 2xy = b \), which leads to the equation \( y^4 + ay^2 - b^2/4 = 0 \). Solving for \( y^2 \) yields
\[
y^2 = \frac{-a \pm \sqrt{a^2 + b^2}}{2} = \frac{|\lambda| - \text{Re } \lambda}{2}
\]
Then the following assertions hold.

**Proposition 3.9.** Suppose that $A$.

**Proof.** In fact, by Lemma 3.7, $\square$

In case b) the estimate is similar and involves (3.6).

The remaining statements follow from the descriptions

$$\lambda \in \partial \Pi_\omega \iff \text{Im } \sqrt{\lambda} = \omega \quad \text{and} \quad \lambda \in \Pi_\omega \iff \text{Im } \sqrt{\lambda} \leq \omega.$$  

Let $\gamma \geq 0$. Then $|\lambda - \gamma| - \Re(\lambda - \gamma) = (|\lambda - \gamma| + \gamma) + (|\gamma - \lambda|)$ and hence

$$(3.6) \quad \left| \text{Im } \frac{\sqrt{\lambda - \gamma}}{\sqrt{\lambda}} \right| \geq \left| \text{Im } \sqrt{\lambda} \right|$$

for any $\lambda \in \mathbb{C}$ by Lemma 3.7. It follows that

$$\gamma + \Pi_\omega \subseteq \Pi_\omega \quad \text{for all } \gamma \geq 0,$$

a fact that is geometrically evident. It follows that if $A$ is of parabola type $\omega$, then $\sigma(A + \gamma) \subseteq \Pi_\omega$. By the following result, $A + \gamma$ is indeed of parabola type $\omega$, too.

**Proposition 3.8.** Let $\omega > 0$ and $\gamma \geq 0$, and define

$$C := \sup \left\{ \left| \frac{\lambda}{\sqrt{\lambda - \gamma}} \right| \mid \lambda \notin \Pi_\omega \right\} < \infty.$$  

Then the following assertions hold.

a) **If** $\|R(\lambda, A)\| \leq \frac{M}{\sqrt{\lambda}}$ **for** $\lambda \notin \Pi_\omega$, **then** $\|R(\lambda, A + \gamma)\| \leq \frac{MC}{\sqrt{\lambda}}$ ($\lambda \notin \Pi_\omega$).

b) **If** $\|R(\lambda, A)\| \leq \frac{M}{\sqrt{\lambda}(\text{Im } \sqrt{\lambda} - \omega)}$ **for** $\lambda \notin \Pi_\omega$, **then**

$$\|R(\lambda, A + \gamma)\| \leq \frac{MC}{\sqrt{\lambda}(\text{Im } \sqrt{\lambda} - \omega)} \quad (\lambda \notin \Pi_\omega).$$

In particular, if for any $\omega \geq 0$, $A \in \text{Para}(\omega)$, then $A + \gamma \in \text{Para}(\omega)$, and if $A \in \text{SPara}(\omega)$, then $A + \gamma \in \text{SPara}(\omega)$.

**Proof.** Fix $\lambda \in \mathbb{C} \setminus \Pi_\omega$. Then $\lambda - \gamma \notin \Pi_\omega$, too. In case a) we can estimate

$$\|R(\lambda, A + \gamma)\| = \|R(\lambda - \gamma, A)\| \leq \frac{M}{\sqrt{\lambda - \gamma}} = \frac{M}{\sqrt{\lambda}} \sqrt{\left| \frac{\lambda}{\lambda - \gamma} \right|} \leq \frac{MC}{\sqrt{|\lambda|}}.$$  

In case b) the estimate is similar and involves (3.6).

In the case $\omega = 0$ we have a better result.

**Proposition 3.9.** Suppose that $\|R(\lambda, A)\| \leq \frac{M}{\sqrt{|\lambda|} \text{Im } \sqrt{\lambda}}$ for all $\lambda \in \mathbb{C} \setminus [0, \infty)$.

Then

$$\|\lambda R(\lambda, A)\| \leq \frac{\sqrt{2}M}{\sqrt{1 - \cos \varphi}} \quad (\varphi = \arg \lambda) \quad \text{for any } \lambda \in \mathbb{C} \setminus [0, \infty)$$

and

$$\|R(\lambda, A)\| \leq \frac{2M}{\text{Im } \lambda} \quad \text{for any } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$  

**Proof.** In fact, by Lemma 3.7,

$$\|R(\lambda, A)\| \leq \frac{M}{\sqrt{|\lambda|} \text{Im } \sqrt{\lambda}} = \frac{M\sqrt{2}}{\sqrt{|\lambda|} \sqrt{|\lambda|} \sqrt{1 - \cos \varphi}}.$$  

which is the first assertion. For the second we simply note that it follows from Lemma 3.7 that
\[ |\text{Im}\, \lambda| = |\text{Im}\, \sqrt{\lambda}| \sqrt{|\lambda|^{2} + 2 \cos \varphi} \leq 2 |\text{Im}\, \sqrt{\lambda}| \sqrt{|\lambda|}. \]

\[ \square \]

Proposition 3.9 says in particular that an operator satisfying its assumptions is of strong strip type 0 and sectorial of angle 0. In general, if we shift a strong parabola type operator far enough to the right, we obtain a sectorial operator (of arbitrary small angle). This is in accordance with the inclusion
\[(\omega/\cos \theta)^2 + \Pi_\omega \subseteq \mathbb{S}_{\pi/2 - \theta} \quad (\omega \geq 0, \theta \in [0, \pi/2)) \]
which is readily established.

**Proposition 3.10.** Let \( \omega \geq 0 \) and let \( A \) be an operator on a Banach space \( X \) such that
\[ \|R(\mu, A)\| \leq \frac{M}{\sqrt{|\mu| (|\text{Im}\, \sqrt{|\mu|} - \omega)}} \quad (\mu \notin \Pi_\omega). \]

Let \( \theta \in [0, \pi/2) \) and define \( B_0 := A + (\omega/\cos \theta)^2 \). Then
\[ \|R(\mu, e^{i\theta} B_0)\| \leq \frac{2M}{\cos \theta |\text{Re}\, \mu|} \quad (\text{Re}\, \mu < 0). \]

In particular, \( B_0 \) is sectorial of angle \( \pi/2 - \theta \) and \( -B_0 \) is of half-plane type 0.

**Proof.** Fix \( \theta \in (-\pi/2, \pi/2) \) and \( \text{Re}\, \mu < 0 \), and let \( \lambda = x + iy \) such that \( y \geq 0 \) and \( \lambda^2 = e^{-i\theta}\mu - (\omega/\cos \theta)^2 \). Since \( e^{-i\theta}\mu \notin \mathbb{S}_{\pi/2 - \theta} \), from (3.7) we conclude that
\[ \lambda^2 \notin \Pi_\omega = \{ z^2 \mid z \in \mathbb{S}_{\omega} \}. \]
It follows that \( \lambda \notin \mathbb{S}_{\omega} \), i.e., \( y > \omega \). By construction,
\[ \mu = e^{i\theta}(\lambda^2 + (\omega/\cos \theta)^2) \] and so
\[ -\frac{\text{Re}\, \mu}{\cos \theta} = y^2 - (\omega/\cos \theta)^2 - x^2 + (\tan \theta)2xy = (y^2 - \omega^2)(1 + \tan^2 \theta) - (x - y \tan \theta)^2. \]
Since \( -\frac{\text{Re}\, \mu}{\cos \theta} > 0 \) by hypothesis, we can estimate
\[ \|R(\mu, e^{i\theta} B_0)\| = \|R(\lambda^2, A)\| \leq \frac{M}{|\lambda| (|\text{Im}\, \lambda - \omega|)} = \frac{M \cos \theta}{|\text{Re}\, \mu|} \sqrt{x^2 + y^2(y - \omega)} \]
\[ = \frac{M \cos \theta}{|\text{Re}\, \mu|} \left( \frac{(y^2 - \omega^2)(1 + \tan^2 \theta) - (x - y \tan \theta)^2}{\sqrt{x^2 + y^2(y - \omega)}} \right) \]
\[ \leq \frac{M \cos \theta}{|\text{Re}\, \mu|} \left( \frac{(y + \omega)(1 + \tan^2 \theta)}{\sqrt{x^2 + y^2}} \right) \leq \frac{2M}{(\cos \theta) |\text{Re}\, \mu|}. \]

The additional assertions follow readily. \( \square \)

**Remark 3.11.** If \(-A\) generates a cosine function \( (C(t))_{t \in \mathbb{R}} \) such that \( \|C(t)\| \leq M_\omega|t|, \) \( t \in \mathbb{R} \), then the better estimate
\[ \|R(\mu, e^{i\theta} B_0)\| \leq \frac{M}{\sqrt{\cos \theta |\text{Re}\, \mu|}} \quad (\text{Re}\, \mu < 0, \theta \in [0, \pi/2)). \]

This can be seen by writing down the holomorphic semigroup generated by \(-A\) using the Gauß–Weierstrass formula, see (4.4) below.

It follows from Propositions 3.8-3.10 that if \( A \in \text{SPara}(\omega) \) and \( B_0 := A + (\omega/\cos \theta)^2 \), then
a) \( B_0 \) is of strong parabola type \( \omega \);
b) \( B_0 \) is of half-plane type \( (\tan \theta)^2 \);
c) \( B_0 \) is sectorial of angle \( \pi/2 - \theta \) (in case that \( \omega > 0 \)).
If we take the square root of $B_\theta$ (defined, for instance, by the holomorphic calculus for sectorial operators) we obtain an operator of strong strip type $\omega$. More precisely, the following holds.

**Proposition 3.12.** Let $\omega \geq 0$ and suppose that $\|R(\lambda, A)\| \leq \frac{M}{\sqrt{\lambda(|\lambda_{\text{Im}}(\omega)\lambda| - \omega)}}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{I}_{\omega}$. Suppose further that $A$ is sectorial and let $A^{1/2}$ be its sectorial square root. Then there is $C \geq 0$ such that

$$\|R(\lambda, A^{1/2})\| \leq \frac{C}{|\lambda_{\text{Im}}(\lambda) - \omega|} \quad \text{for all } \lambda \notin \mathbb{S}_{\omega}.$$ 

**Proof.** Write $B := A^{1/2}$ and let $\lambda \in \mathbb{C} \setminus \mathbb{S}_{\omega}$. Then

$$(\lambda^2 - A) = - (\lambda - B)(-\lambda - B).$$

It follows that $\pm \lambda \in \rho(B)$ with

$$R(\lambda^2, A) = - R(\lambda, B)R(-\lambda, B) = \frac{1}{2\lambda}(R(\lambda, B) - R(-\lambda, B)).$$

From the theory of sectorial operators it is known that $B$ is sectorial of angle $< \pi/2$ [10, Prop.3.1.2]. Hence,

$$C' := \sup\{\|\lambda R(\lambda, B)\| \mid \text{Re } \lambda < 0\} < \infty,$$

and therefore

$$\|R(\lambda, B)\| \leq \frac{C'}{|\lambda|} \leq \frac{C'}{|\lambda_{\text{Im}}(\lambda) - \omega|} \quad \text{if Re } \lambda \leq 0, |\lambda_{\text{Im}}(\lambda) > \omega.}$$

If Re $\lambda > 0$, then Re $(-\lambda) < 0$, and hence

$$\|R(\lambda, B)\| \leq \frac{2M}{|\lambda_{\text{Im}}(\lambda) - \omega|} + \frac{C'}{|\lambda_{\text{Im}}(\lambda) - \omega|},$$

and the claim is established.

**Remark 3.13.** Proposition 3.12 is due to Vörös [22, Cor. 5.14]. Our proof is based on [14, Lemma 6.1], which covered a special case.

Back to Proposition 3.10, $B_\theta^{1/2}$ is both sectorial of angle $\pi/4 - \theta/2$ and of strong strip type $\omega$. One can easily show that a composition rule

$$f(B_\theta) = [f(z + (\omega/\cos \theta)^2)](A) = [f(z^2)](B_\theta^{1/2})$$

holds when $f(B_\theta)$ is defined by the functional calculus for sectorial operators (or also, in the case $\theta = 0$, by the one for operators of half-plane type). By [10, Prop. 1.3.6] one needs to know it only for functions $f$ belonging to a generating set for the primary calculus for $B_\theta$. Here the assertion follows from Cauchy’s theorem.

3.4. **Operator Cosine Functions.** How can we access operator cosine functions by the functional calculus? Let $A$ be of weak parabola type $\omega \geq 0$. Consider the functions

$$c_t(z) := \cos(t\sqrt{z}) \quad (t \in \mathbb{R})$$

which are bounded holomorphic functions on every parabola $\Pi_\varphi$, $\varphi > 0$. So

$$C_A(t) := c_t(A) \quad (t \in \mathbb{R})$$

is a well defined family of closed operators. Since for $\mu > \omega^2$ the function $(\mu + z)^{-2}$ is a regularizer for $c_t$,

$$\text{dom}(A^2) \subseteq \text{dom}(C_A(t))$$

for all $t \in \mathbb{R}$. If $A$ is of parabola type $\omega \geq 0$, then by Remark 3.5 $c_t$ is regularized by $(\mu + z)^{-1}$, and hence $\text{dom}(A) \subseteq \text{dom}(C_A(t))$. 


The following shows that our definition of $C_A(t)$ is coherent with the classical one via Laplace transforms, and establishes the link with the complex inversion formula.

**Lemma 3.14 (Complex Inversion Formula).** Let $A$ be an operator of (weak) parabola type $\omega \geq 0$ on the Banach space $X$, and let $\varphi > \omega$. Then for $x \in \text{dom}(A)$ ($x \in \text{dom}(A^2)$) the function

$$(t \mapsto C_A(t)x) : \mathbb{R} \to X$$

is continuous and even and satisfies $\sup_{t \in \mathbb{R}} \|e^{-|t|}C_A(t)x\| < \infty$. Its Laplace transform is

$$\int_0^\infty e^{-\lambda t}C_A(t)x \, dt = \lambda R(\lambda^2, -A)x \quad (\text{Re } \lambda > \varphi).$$

Moreover, $C_A(t)x$ is also given by the improper integrals

$$(3.9) \quad C_A(t)x = \frac{1}{2\pi i} \int_{\partial \Pi_\varphi} \cos(t\sqrt{z})R(z, A)x \, dz \quad (t \in \mathbb{R})$$

$$(3.10) \quad = \frac{1}{2\pi i} \int_{\text{Re } z = \varphi} e^{t\bar{z}}R(z^2, -A)x \, dz \quad (t > 0).$$

**Proof.** Let $d := 2$ if $A$ is of weak parabola type, and $d := 1$ if $A$ is of parabola type $\omega$. Fix $\mu > \omega^2$ and $x \in \text{dom}(A^d)$, and let $x_j := (\mu + A)^{-1}x$ for $1 \leq j \leq d$. Then

$$C_A(t)x = \left(\frac{\cos(t\sqrt{z})}{(\mu + z)^d}\right)(A)x_d = -\frac{1}{\pi i} \int_{\text{Im } z = \varphi} \cos(tz)\frac{R(z^2, A)x_d}{(\mu + z)^d} \, dz.$$ 

Therefore,

$$\|C_A(t)x\| \lesssim e^{\varphi|t|} \int_{\text{Im } z = \varphi} \frac{|R(z^2, A)|}{|\mu + z|^d} \|dz\| \|x_d\|.$$ 

The continuity of $t \mapsto C_A(t)x$ follows from Lebesgue’s theorem. Taking Laplace transforms we obtain

$$\int_0^\infty e^{-\lambda t}C_A(t)x \, dt = \frac{1}{2\pi i} \int_{\partial \Pi_\varphi} \left(\int_0^\infty e^{-\lambda t}\cos(t\sqrt{z}) \, dt\right) \frac{1}{(\mu + z)^d} R(z, A) \, dz \, x_d$$

$$= \frac{1}{2\pi i} \int_{\partial \Pi_\varphi} \lambda (\mu + z)^d R(z, A) \, dz \, x_d$$

$$= \lambda(\mu + A)^{-1} \int_{\partial \Pi_\varphi} R(z, A) \, dz \, x_d = \lambda R(\lambda^2, -A)x$$

for $\text{Re } \lambda > \varphi$. To establish the remaining part, first note that

$$(3.11) \quad \int_{\partial \Pi_\varphi} \frac{\cos(t\sqrt{z})}{\mu + z} \, dz = 0$$

in the improper sense. This follows from Cauchy’s theorem and the fact that the lengths of the line segments $\{\text{Re } z = N\} \cap \Pi_\varphi$ grow as $\sqrt{N}$, whereas the integrand is $O(1/N)$ on these segments.

We begin with the weak parabola case ($d = 2$). With $x_1, x_2$ being defined as above we have $R(z, A)x_j = (\mu + z)R(z, A)x_{j-1} - x_{j-1}$ for $j = 1, 2$. Hence,

$$C_A(t)x = \frac{1}{2\pi i} \int_{\partial \Pi_\varphi} \frac{\cos(t\sqrt{z})}{(\mu + z)^2} R(z, A)x_2 \, dz$$

$$= \frac{1}{2\pi i} \int_{\partial \Pi_\varphi} \frac{\cos(t\sqrt{z})}{\mu + z} R(z, A)x_1 \, dz - \frac{1}{2\pi i} \int_{\partial \Pi_\varphi} \frac{\cos(t\sqrt{z})}{(\mu + z)^2} \, dz \, x_1$$

$$= \frac{1}{2\pi i} \int_{\partial \Pi_\varphi} \frac{\cos(t\sqrt{z})}{\mu + z} R(z, A)x_1 \, dz,$$
and this last integral is absolutely convergent. (Note that the function \( \cos(t\sqrt{z})(\mu + z)^{-2} \) is an element of \( \mathcal{E}(\Pi_\omega) \) for any \( \varphi > 0 \).) Then, by (3.11) and again the resolvent formula from above, we can continue

\[
C_A(t)x = \frac{1}{2\pi i} \int_{\partial \Pi_\varphi} \frac{\cos(t\sqrt{z})}{\mu + z} R(z, A)x \, dz
\]

\[
= \frac{1}{2\pi i} \int_{\partial \Pi_\varphi} \cos(t\sqrt{z})R(z, A)x \, dz - \frac{1}{2\pi i} \int_{\partial \Pi_\varphi} \frac{\cos(t\sqrt{z})}{\mu + z} \, dz \, x
\]

\[
= \frac{1}{2\pi i} \int_{\partial \Pi_\varphi} \cos(t\sqrt{z})R(z, A)x \, dz.
\]

(Note that whereas the first integral is absolutely convergent, the following converge only in the improper sense.) This settles (3.9) for the weak parabola type case. In the parabola type case we can directly start with the second computation, and hence (3.9) is established in both cases.

For the remaining part we make a change of variable \( z \mapsto (iz)^2 = -z^2 \), with \( \text{Re } z = \varphi \) and obtain

\[
C_A(t) = \frac{1}{2\pi i} \int_{\text{Re } z = \varphi} \cos(tiz)2zR(z^2, -A)x \, dz
\]

\[
= \frac{1}{2\pi i} \int_{\text{Re } z = \varphi} (e^{tz} + e^{-tz})zR(z^2, -A)x \, dz.
\]

Note that \( zR(z^2, -A)x = \frac{1}{\sqrt{2\pi}}x - \frac{1}{\sqrt{2\pi}}zR(z^2, -A)Ax \) and hence

\[
\int_{\text{Re } z = \varphi} e^{-tz}zR(z^2, -A)x \, dz = \int_{\text{Re } z = \varphi} \frac{e^{-tz}}{z} \, dz \, x - \int_{\text{Re } z = \varphi} \frac{e^{-zt}}{z^2} zR(z^2, -A)x \, dz,
\]

which is equal to 0 by Cauchy’s theorem. Hence we conclude that (3.10) is true. □

We remark that (3.10) expresses that in case \( x \in \text{dom}(A) \) and \( x \in \text{dom}(A^2) \), respectively, the complex inversion formula holds for the Laplace-transform pair \( (C_A(t)x, zR(z^2, -A)x) \).

**Theorem 3.15 (Generation).** Let \( A \) be an operator of weak parabola type \( \omega \geq 0 \) on the Banach space \( X \). Then \( -A \) generates a cosine function \( (C(t))_{t \in \mathbb{R}} \) if and only if \( A \) is densely defined and \( C_A(t) \) (defined by (3.8)) is a bounded operator for each \( t \in \mathbb{R} \). In this case \( C(t) = C_A(t), \ t \in \mathbb{R} \).

**Proof.** If \( -A \) generates a cosine function \( (C(t))_{t \in \mathbb{R}} \), then \( \text{dom}(A) \) is dense. A standard argument then shows that \( \text{dom}(A^2) \) is dense as well. Moreover, by the uniqueness of Laplace transforms, \( C(t)x = C_A(t)x \) for each \( x \in \text{dom}(A^2) \). Since \( C_A(t) \) is a closed operator it follows that \( C_A(t) = C(t) \).

Conversely, suppose that \( C_A(t) \) is a bounded operator for each \( t \in \mathbb{R} \), and that \( \text{dom}(A) \) (and hence \( \text{dom}(A^2) \)) is dense. Then the operator-valued mapping

\[
\mathbb{R} \rightarrow \mathcal{L}(X), \quad t \mapsto C_A(t)
\]

is strongly measurable and satisfies “d’Alembert’s identity” (1.1). By a standard result [6, Lemma 5.2] it follows that \( C_A(t) \) is strongly continuous, whence it is a cosine function. By Lemma 3.14 and the density of \( \text{dom}(A) \) again, \( -A \) is its generator. □

As a corollary we improve on a recent result of Król [18, Thm. 2.6, (ii)⇒(i)]. The proof is just a combination of Lemma 3.14 and Theorem 3.15.

**Corollary 3.16.** Let \( A \) be a densely defined operator of parabola type \( \omega \geq 0 \) on some Banach space \( X \) and let \( \varphi > \omega \). If the operators

\[
\int_{\varphi - ia}^{\varphi + ia} e^{tz}zR(z^2, -A) \, dz \quad (a > 0)
\]
are uniformly bounded for each \( t > 0 \), then \(-A\) is the generator of a cosine function.

The converse of Corollary 3.16 holds on UMD spaces, see [13, Thm. 4.2] and cf. Kröl’s account in [18, Lemma 4.3]. Hence we can state the following nice characterization of cosine function generators on UMD spaces.

**Corollary 3.17.** Let \( A \) be an operator of weak parabola type \( \omega \) on a UMD space \( X \). Then the following assertions are equivalent:

(i) \(-A\) generates an operator cosine function.

(ii) The operators

\[
\int_{\varphi - ia}^{\varphi + ia} e^{zt} zR(z^2, -A) \, dz \quad (a > 0)
\]

are uniformly bounded for each \( t > 0 \) and one/all \( \varphi > \omega \).

(iii) For each \( t > 0 \) and \( x \in X \) the improper integral

\[
\int_{\varphi - i\infty}^{\varphi + i\infty} e^{zt} zR(z^2, -A)x \, dz
\]

exists.

**Remarks 3.18.**

1) The hypothesis of Corollary 3.16 is satisfied, for example, if

\[
\lim_{a \to \infty} \int_{\varphi - ia}^{\varphi + ia} e^{zt} \left< zR(z^2, -A)x, x' \right> \, dz
\]

exists for each \( x \in X, x' \in X' \).

This is Kröl’s condition in [18, Thm. 2.6 (ii)]. Although Kröl has an additional condition, he concedes in [18, Rem. 3.3a] that (3.12) is the essence of the matter. Our Corollary 3.16 is a slight improvement as the uniform boundedness of the partial integral operators is weaker than (3.12) by the uniform boundedness theorem.

2) In [18, Rem. 3.3] Kröl mentions that even the weak parabola type condition is of a technical nature and can be weakened. Indeed, any polynomial growth of the resolvent outside a parabola \( \Pi_\omega \) can be allowed. One simply has to adjust the class of elementary functions in the definition of the functional calculus to compensate for that growth. Bounded holomorphic functions can be regularized by sufficiently high powers of the resolvent, and Theorem 3.14 holds for such operators and \( x \in \text{dom}(A^d) \) for sufficiently large \( d \in \mathbb{N} \). As these subspaces are still dense, also Corollary 3.16 holds for such operators.

3) Using a Convergence Lemma for the functional calculus on parabolas analogous to the one on sectors [10, Sec.5.1] and on halfplanes [3], one can obtain (rational) approximation results for cosine functions. Similarly, a Trotter-Kato type result as in [3, Sec.4] holds for cosine functions, see [7].

4. **The Hille-Phillips Calculus for Cosine Functions**

The holomorphic functional calculus discussed in the previous section presupposes only knowledge about the resolvent of an operator, and hence can be used to decide whether an operator generates a cosine function or not. In contrast to this, if one already knows that this is the case, one can extend the holomorphic calculus in a natural way.

To make this precise, suppose that \(-A\) generates the strongly continuous cosine function \((C(s))_{s \in \mathbb{R}}\) satisfying the exponential growth condition

\[
\|C(s)\| \leq Me^{\omega |s|} \quad (s \in \mathbb{R}).
\]
If \( \mu \) is a Borel measure on \( \mathbb{R} \) satisfying \( \int_{\mathbb{R}} e^{i|z|} |\mu| (ds) < \infty \) then one can form the strong integral

\[
C_\mu := \int_{\mathbb{R}} C(s) \mu(ds).
\]

Since the cosine function is even, one has \( C_\mu = C_{\mu_e} \) where

\[
\mu_e(B) := \frac{1}{2} (\mu(B) + \mu(-B)) \quad (B \text{ a Borel subset of } \mathbb{R})
\]

is the even part of \( \mu \). A measure \( \mu \) is even if \( \mu = \mu_e \). Let

\[
M_\omega(\mathbb{R}) := \left\{ \mu \in M(\mathbb{R}) \mid \int_{\mathbb{R}} e^{i|z|} |\mu| (ds) < \infty \right\}
\]

due to the norm

\[
\|\mu\|_{M_\omega} := \int_{\mathbb{R}} e^{i|z|} |\mu| (ds).
\]

Then it is easy to see that \( M_\omega(\mathbb{R}) \) is a Banach algebra with respect to convolution, the even measures \( M^e_\omega(\mathbb{R}) := \{ \mu \in M_\omega(\mathbb{R}) \mid \mu = \mu_e \} \) form a closed subalgebra, and

\[
M^e_\omega(\mathbb{R}) \rightarrow \mathcal{L}(X), \quad \mu \mapsto C_\mu
\]

is a homomorphism of algebras satisfying the “transference estimate”

\[
\|C_\mu\| \leq M \|\mu\|_{M_\omega} \quad (\mu \in M^e_\omega(\mathbb{R})),
\]

cf. [14, Prop.2.1]. This homomorphism is called the Hille-Phillips (functional) calculus for the given cosine function.

In order to transform this into a functional calculus for the operator \( A \), we look at the corresponding operation on the level of functions. For \( \mu \in M_\omega(\mathbb{R}) \) its cosine transform is

\[
\hat{\mu}(z) := \int_{\mathbb{R}} \cos(sz) \mu(ds) \quad (|\text{Im } z| \leq \omega).
\]

Then \( \hat{\mu} = \hat{\mu}_e \) and this coincides with the Fourier transform of \( \mu_e \). The function \( g := \hat{\mu} \) is bounded and continuous on the vertical strip \( \operatorname{St}_\omega \), and holomorphic in its interior. Moreover, \( g \) is an even function, and hence the function

\[
f : \overline{\Pi_\omega} \rightarrow \mathbb{C}, \quad z \mapsto f(z) := g(\sqrt{z})
\]

is well defined, bounded, and holomorphic in the interior of \( \Pi_\omega \). We define

\[
f(A) := C_\mu \quad \text{if} \quad f(z) = \hat{\mu}(\sqrt{z}), \quad \mu \in M^e_\omega(\mathbb{R}).
\]

This is a good definition since the Fourier transform is injective. The space

\[
\mathcal{F}(\Pi_\omega) := \{ \hat{\mu}(\sqrt{z}) \mid \mu \in M^e_\omega(\mathbb{R}) \}
\]

is an algebra with respect to pointwise multiplication and the mapping

\[
\Phi : \mathcal{F}(\Pi_\omega) \rightarrow \mathcal{L}(X), \quad f = \hat{\mu}(\sqrt{z}) \mapsto f(A) = C_\mu = \int_{\mathbb{R}} C(s) \mu(ds)
\]

is a homomorphism of algebras. This is called the Hille-Phillips (functional) calculus for \( A \).

**Remark 4.1.** The Hille-Phillips calculus for bounded cosine functions \( (\omega = 0) \) has been introduced already in [14]. The extension to \( \omega > 0 \) is similar to the step from bounded groups to unbounded groups in [12].
4.1. A Special Case. In the next step we consider a particular instance of the Hille-Phillips calculus. The space

\[ X := L^1_1(\mathbb{R}) := \{ f \in L^1(\mathbb{R}) \mid \|f\|_\omega := \int_\mathbb{R} |f(t)| e^{\omega|t|} \, dt < \infty \} \]

is a Banach space with respect to the norm \( \|\cdot\|_\omega \). On \( X \) we consider the translation group defined by

\[ [U(s)f](t) := f(t-s) \quad (s,t \in \mathbb{R}, f \in L^1_1(\mathbb{R})). \]

Then \( U \) is a strongly continuous group with generator \(-d/dt\) with its natural domain, cf. [10, Sec.8.4]. The corresponding cosine function is \( C(s) := \frac{1}{2}(U(s) + U(-s)) \), i.e.,

\[ [C(s)f](t) = \frac{1}{2}(f(t-s) + f(t+s)) \quad (s,t \in \mathbb{R}, f \in L^1_1(\mathbb{R})), \]

with generator being the one-dimensional Laplacian \( \Delta u = u'' \) on its natural domain. If \( \mu \in M_c(\mathbb{R}) \) is even, then it is easy to see that

\[ C_\mu f = \mu * f, \]

i.e., \( C_\mu \) is simply convolution with \( \mu \). The following sharpens the functional calculus estimate (4.2) in the given example.

**Proposition 4.2.** In the example sketched above one has

\[ \|C_\mu\| = \sup\{\|\mu * f\|_\omega \mid \|f\|_\omega \leq 1\} = \|\mu\|_{M_\omega} \]

for every even measure \( \mu \in M_c(\mathbb{R}) \). In other words, the Hille-Phillips functional calculus

\[ \Psi : M^e_\mu(\mathbb{R}) \to \mathcal{L}(X), \quad \mu \mapsto C_\mu \]

is an isometric homomorphism of Banach algebras.

Only the isometry property is to show, and this has been done in [12, p.535].

4.2. Compatibility of the Functional Calculi. As before, let \(-A\) generate the cosine function \((C(s))_{s \in \mathbb{R}}\) on a Banach space \( X \), with the growth estimate (4.1). As explained in (3.1) we then have the estimate

\[ \|R(-\lambda^2, A)\| \leq \frac{M}{|\lambda| (\text{Re} \lambda - \omega)} \quad (\text{Re} \lambda > \omega) \]

and \( A \) is of strong parabola type \( \omega \). As such, the holomorphic functional calculus for \( A \) constructed in Section 3 is at hand, with its elementary class \( \mathcal{E}(\Pi_{\omega}) \) as defined in Remark 3.5. The straightforward question arises how the holomorphic and the Hille-Phillips calculus relate. The answer is simple and given in the following result.

**Theorem 4.3.** Let \(-A\) be the generator of a cosine function \((C(s))_{s \in \mathbb{R}}\) as above, let \( \varphi > \omega \) and \( f \in \mathcal{E}(\Pi_{\omega}) \). Then there is \( g \in L^1_1(\mathbb{R}) \) with \( f(z) = \hat{g}(\sqrt{z}) \) for \( z \in \Pi_{\omega} \), and the two definitions of “\( f(A) \)” (via holomorphic or Hille-Phillips calculus) lead to the same operator.

**Proof.** Let \( \lambda \in C \setminus \Pi_{\omega} \) and let \( w^2 = -\lambda \) with \( \text{Re} w > \omega \). Then

\[ R(\lambda, A) = -R(-\lambda, -A) = -\frac{1}{w} \int_0^\infty e^{-ws} C(s) \, ds. \]

If we apply this to the special case \( A_0 = -\Delta \) on \( X_0 = L^1_1(\mathbb{R}) \) as above, we hence obtain an even measure \( \nu_\lambda \in M^e_c(\mathbb{R}) \) with \( R(\lambda, A_0)g = \nu_\lambda * g \) for \( g \in X_0 \). Indeed, \( \nu_\lambda \) has the density \( g_\lambda(s) = (-1/2w)e^{-ws} \) with respect to Lebesgue measure, and

\[ \hat{\nu}_\lambda(\sqrt{z}) = -\frac{1}{w} \int_0^\infty e^{-ws} \cos(s\sqrt{z}) \, ds = \frac{1}{\lambda - \sqrt{z}^2} = (\lambda - z)^{-1}. \]
We apply the holomorphic functional calculus to \( A_0 \) and obtain

\[
f(A_0) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R(\lambda, A_0) \, d\lambda.
\]

Note that this integral converges in the operator norm. Hence, by Proposition 4.2, \( f(A_0) \) is convolution with

\[
\mu_f := \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) \nu_\lambda \, d\lambda.
\]

As all \( \nu_\lambda \) are contained in \( L^1_c(\mathbb{R}) \), so must \( \mu_f \), i.e., there is \( g \in L^1_c(\mathbb{R}) \) with \( g(s)ds = \nu_f \). By construction

\[
\tilde{\mu}_f(z) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) \tilde{\nu}_\lambda(z) \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) (\lambda - z)^{-1} \, d\lambda = f(z).
\]

This shows that \( f \big|_{\Pi_s} \in \mathcal{F}(\Pi_s) \). Finally,

\[
f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R(\lambda, A) \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) \int_{\mathbb{R}} C(s) \nu_\lambda(s) \, ds \, d\lambda
\]

\[
= \int_{\mathbb{R}} C(s) \left( \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) \nu_\lambda \, d\lambda \right) \, ds = \int_{\mathbb{R}} C(s) \mu_f(ds)
\]

as claimed. \( \square \)

\textbf{Remark 4.4.} Theorem 4.3 essentially says that if \( f \in \mathcal{E}(\Pi_\varphi) \) for some \( \varphi > \omega \geq 0 \), then there is some \( \mu \in \mathcal{M}_c(\mathbb{R}) \) with \( \hat{\mu}(z) = f(z^2) \). This is a statement about \textit{Fourier inversion}, and (4.3), which expresses \( \mu \) in terms of the known measures \( \nu_\lambda = (-1/2w) e^{-w|s|}(ds) \), is a (complex) Fourier inversion formula.

On the other hand, if one takes Fourier inversion for granted, then one can give an alternative proof of Theorem 4.3 along the lines of [12, Lemma 2.2]: Given \( f \in \mathcal{E}(\Pi_\varphi) \) as before and \( \omega < \omega' < \varphi \) one finds by standard Fourier inversion as in [12, Lemma 2.2] a function \( h \in L^1_c(\mathbb{R}) \) such that \( \hat{h}(w) = f(w^2) \) for \( |\text{Im } w| \leq \omega' \). Without loss of generality one may suppose that \( h \) is even. Using the formula

\[
R(z, A) = \frac{1}{\pi} \int_{\mathbb{R}} e^{iws} C(s) \, ds \text{ if } \text{Im } w = \omega' \text{ and } w^2 = z
\]

we compute

\[
f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z) R(z, A), dz = \int_{\mathbb{R} + i\omega'} \hat{h}(w) \frac{1}{\pi} \int_{-\infty}^{\infty} e^{iws} C(s) \, ds \, dw
\]

\[
= 2 \int_{0}^{\infty} \frac{1}{2\pi} \int_{\mathbb{R} + i\omega'} e^{iws} \hat{h}(w) \, dw \, C(s) \, ds = 2 \int_{0}^{\infty} \frac{1}{2\pi} \int_{\mathbb{R}} e^{iws} \hat{h}(w) \, dw \, C(s) \, ds
\]

\[
= 2 \int_{\infty}^{\infty} h(s) C(s) \, ds = \int_{\mathbb{R}} C(s) (h(s)ds),
\]

where we used path deformation and the Fourier inversion formula.

\textbf{4.3. The Gauss-Weierstrass Formula.} Each negative of a strong parabola type operator generates a holomorphic semigroup. This follows from Proposition 3.10, but can be seen directly via the holomorphic functional calculus. Indeed, for \( \Re \tau > 0 \) the function \( z \mapsto e^{-\tau z} \) is contained in \( \mathcal{E}(\Pi_\varphi) \) for each \( \varphi > 0 \), and hence \( e^{-\tau A} := (e^{-\tau z})(A) \) is defined by the holomorphic calculus for any (weak) parabola type operator \( A \). The holomorphy of the function \( \tau \mapsto e^{-\tau A} \) is easily seen by looking at the Cauchy integrals. Finally, to prove the uniform boundedness of the operators \( e^{-\tau A} \) as \( \tau \to 0 \) in sectors \( |\arg \tau| \leq \eta < \pi/2 \) one has to use the resolvent estimate of strong parabola type operators and choose the integration contour as \( \partial \Pi_\varphi \) with \( \varphi = |\Re \tau|^{-1/2} \).

Suppose that \( -A \) generates a cosine function \((C(s))_{s \in \mathbb{R}}\). By Theorem 4.3 and Remark 4.4, there is some \( \mu_\tau \in \mathcal{M}_c(\mathbb{R}) \) with

\[
e^{-\tau A} = \int_{\mathbb{R}} C(s) \mu_\tau(ds) \quad (\Re \tau > 0).
\]
Since this must work for every cosine function, \( \omega \) can be chosen arbitrary large. Hence \( \mu_\tau \) must satisfy

\[
\hat{\mu}_\tau(z) = e^{-\tau z^2}
\]

for all \( z \in \mathbb{C} \), and it is known from Fourier analysis that

\[
\mu_\tau = \frac{e^{-\tau z^2/4}}{\sqrt{4\pi \tau}} ds.
\]

As a consequence one obtains the Gauss-Weierstrass formula [1, (3.102)]

\[
e^{-\tau A} = \frac{1}{\sqrt{\pi \tau}} \int_0^\infty e^{-s^2/4\tau} C(s) ds.
\]

5. A Transference Principle for Cosine Functions

The estimate (4.2) together with Proposition 4.2 translates into the statement that in general \( \|f(A)\| \) can be estimated in terms of the norm of a convolution operator. Moreover, Proposition 4.2 tells that this is the best possible estimate in general. However, in special situations better estimates are possible.

A transference principle factors the operator \( f(A) \) over a convolution operator on a possibly vector-valued function space. As a consequence one obtains estimates in terms of these convolution operators and they may happen to be much better than the universal estimate (4.2).

The underlying technique to establish such a factorization is actually quite old and goes back at least to Hille and Phillips. In the context of ergodic theory it was put forward by Calderón, then generalized by Coifman and Weiss in the 1970’s and further by Berkson, Gillespie and Muhly in [4]. Recently, the general scheme behind transference has been brought to light and that has been fruitfully applied to strongly continuous groups [12, 15], bounded cosine functions [14] and finally to general strongly continuous semigroups [17]. See these papers for more information on transference and its applications.

In the present section we want to extend the list of instances of transference principles to include general (=unbounded) cosine functions. On first glance this seems to be superfluous, for the following reason.

Let \( -A \) be the generator of a cosine function \( (C(s))_{s \in \mathbb{R}} \) satisfying \( \|C(s)\| \leq M e^{\omega|s|} \) as above. Consider the operator matrix

\[
A = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix};
\]

then \( -iA \) generates a group \( (\mathcal{U}(s))_{s \in \mathbb{R}} \) on the space \( X = V \times X \), where \( V \) is the so-called Kisyński space

\[
V := \{ x \in X \mid t \mapsto S(t)x \in C([0,1]; \text{dom}(A)) \},
\]

with the norm \( \|x\|_V = \|x\| + \sup_{0 \leq s \leq 1} \|AS(s)x\| \), where

\[
S(t) := \int_0^t C(s) ds \quad (t \in \mathbb{R})
\]

is the associated sine function. The group \( \mathcal{U}(t) \) is given by

\[
\mathcal{U}(t) = \begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{pmatrix} \quad (t \in \mathbb{R}).
\]

These results are standard, see e.g. [1, Thm. 3.14.11]. It has been proved in [12] that the group type of \( \mathcal{U} \) coincides with the exponential growth bound of the original cosine function. Hence one can apply the result of [15] to the group \( \mathcal{U} \), and projecting onto the first coordinate yields a transference principle for the original cosine function.

However, one may wonder whether one really needs the detour via the group \( \mathcal{U} \), and it is the purpose of this section to show that one can indeed arrive at the
transference principle directly. Moreover, the statement about the group type of $U$
mentioned above was proved in [12] in a quite awkward way, and we shall present
a much shorter and more elegant proof here, with an even slightly sharper result.
As a byproduct we derive some useful results about the Kisyński space. We try to
keep the section self-contained.

5.1. Cosine Function Identities. From the defining functional equations

$$2C(t)C(s) = C(t - s) + C(t + s), \quad C(0) = I, \quad S(t) = \int_0^t C(s) \, ds$$

one arrives by simple computation at

$$(5.4) \quad 2S(t)C(s) = S(t - s) + S(t + s),$$

$$(5.5) \quad S(t + s) = S(t)C(s) + S(s)C(t).$$

These identities imply that $s \mapsto C(s)S(t)$ is differentiable and

$$(5.6) \quad C(t)C(s) + \frac{d}{ds} C(s)S(t) = C(t + s),$$

$$(5.7) \quad \frac{d}{ds} C(s)S(t) = \frac{d}{dt} C(t)S(s).$$

Define $H(t) := \int_0^t S(s) \, ds = \int_0^t (t-s)C(s) \, ds$. Then we have

$$C(t) = c_t(A) \quad \text{where} \quad c_t(z) = \cos(t\sqrt{z}),$$

$$S(t) = s_t(A) \quad \text{where} \quad s_t(z) = \frac{\sin(tw)}{w|_{w=\sqrt{z}},}$$

$$H(t) = h_t(A) \quad \text{where} \quad h_t(z) = \frac{1 - \cos(t\sqrt{z})}{z}.$$ 

In particular we find that $zh_t(z) = 1 - c_t(z)$, and hence $\text{ran}(H(t)) \subseteq \text{dom}(A)$ with $AH(t) = I - C(t)$. (This is [1, Prop. 3.14.5a]; note that here $-A$ is the generator
and not $A$.) Integrating (5.5) with respect to $s$ we find

$$H(t + s) - H(t) = S(t)S(s) + H(s)C(t),$$

which implies that $S(t)S(s)$ maps $X$ into $\text{dom}(A)$, and

$$AS(t)S(s) = I - C(t + s) - (I - C(t)) - (I - C(s))C(t) = C(s)C(t) - C(t + s).$$

Hence we arrive at the important decoupling identity

$$(5.8) \quad C(t + s) = C(t)C(s) - AS(t)S(s) \quad (s, t \in \mathbb{R}),$$

unfortunately missing in the otherwise excellent reference [1]. (However, it is of
course a straightforward consequence of [1, Prop. 3.14.5f]; it is known for a long
time, see e.g., [20, Lemma 2.2d].)

Finally, change $s$ for $-s$ in (5.8) and subtract the two identities to obtain

$$(5.9) \quad -2AS(t)S(s) = C(t + s) - C(t - s) \quad (s, t \in \mathbb{R}),$$

which is [1, Prop. 3.14.5f]. In particular,

$$(5.10) \quad \|AS(t)S(s)\| \leq Me^{-|t|}e^{-|s|} \quad (s, t \in \mathbb{R}).$$

5.2. The Kisyński Space. The Kisyński space has been defined in (5.2). We
employ an idea of [12, Lemma 6.1]. Putting $s = t$ and integrating over $[0,1/2]$ in
(5.9) we obtain

$$(5.11) \quad I = S(1) + 4 \int_0^{1/2} AS(s)S(s) \, ds$$

This leads to the following central result.
Theorem 5.1. Let \(-A\) be the generator of a cosine function \((C(s))_{s \in \mathbb{R}}\) satisfying \(\|C(s)\| \leq M e^{\omega|s|}\) for \(s \in \mathbb{R}\). Let also \(p \in [1, \infty)\). Then the following assertions hold.

a) \(S(t)\) maps \(X\) into \(V\) with \(\|S(t)\|_{X \to V} \lesssim \begin{cases} e^{|t|} & (t \in \mathbb{R}) \quad \text{if } \omega > 0; \\ |t| & (t \in \mathbb{R}) \quad \text{if } \omega = 0. \end{cases}\)

b) \(S(t)\) maps \(V\) into \(\text{dom}(A)\) with \(\|AS(t)\|_{V \to X} \lesssim e^{|t|}\) \((t \in \mathbb{R})\).

c) One has \(x \in V\) if and only if \(s \mapsto AS(s)x \in L^p((0,1);X)\); moreover
\[
\|x\|_V \sim \left(\|x\|_X^p + \int_0^1 \|AS(s)x\|^p_X \, ds\right)^{1/p}.
\]

Proof. a) follows directly from (5.10). For b) let \(x \in V\). Then by (5.11) we have
\[
S(t)x = S(t)S(1)x + 4 \int_0^{1/2} S(t)S(s)AS(s)x \, ds \quad (t \in \mathbb{R}).
\]
This shows that \(S(t)x \in \text{dom}(A)\). Moreover, by (5.10) again,
\[
\|AS(t)x\| \leq \|AS(t)S(1)x\| + 2 \int_0^{1/2} \|AS(t)S(s)\| \|AS(s)x\| \, ds \leq 2M e^{2|t|} \|x\|_V.
\]
Finally, the previous shows that for \(0 \leq t \leq 1\),
\[
\|AS(t)x\| \leq M e^{2\omega t} \|x\| + 2M e^{2\omega t} \|s \mapsto AS(s)\|_{L^p((0,1);X)}.
\]
This establishes the claimed equivalence of norms for any \(p\), and c) follows. \(\square\)

Part c) of Theorem 5.1 has been established already in [12]. As remarked there, it shows that the space \(V\) inherits geometric properties from \(X\): if \(X\) is a Hilbert (or UMD or subspace of an \(L^p\)-) space then so is \(V\). However, the estimates in parts a) and b) of Theorem 5.1 go beyond the results of [12]. They account for the following theorem.

Theorem 5.2. Let \(-A\) be the generator of a cosine function \((C(s))_{s \in \mathbb{R}}\) on the Banach space \(X\) satisfying \(\|C(s)\| \leq M e^{\omega|s|}\) for \(s \in \mathbb{R}\). If \(\omega > 0\) then there is \(M' \geq 0\) such that
\[
\|U(s)\| \leq M' e^{\omega|s|} \quad (s \in \mathbb{R}),
\]
where \(U\) is the group on \(V \times X\) generated by \(-iA\) and \(A\) is as in (5.1).

Proof. This follows from (5.3) and Theorem 5.1, part a) and b). \(\square\)

Remark 5.3. In [12] it was proved (by a quite involved argument retracing the whole construction in [1] of the group \(U\)) that the group type of \(U\) equals the exponential growth bound of \((C(s))_{s \in \mathbb{R}}\). As the group type of \(U\) is the infimum of those \(\omega' > 0\) such that there is \(M' \geq 1\) with \(\|U(s)\| \leq M' e^{\omega|s|}\omega'\) for all \(\omega\), this is a consequence of Theorem 5.2, which is a strictly sharper in the case \(\omega > 0\).

Unfortunately, for \(\omega = 0\) the group \(U\) doesn’t seem to be bounded any more. This is due to the \(\|\cdot\|_X\)-part of the norm of \(V\). As a consequence one cannot get a sharp transference principle for bounded cosine functions by passing to the group \(U\). In fact, one has to go a different route, see [14].

5.3. Transference Principles for Cosine Functions. Let as before \(-A\) be the generator of a cosine function \((C(s))_{s \in \mathbb{R}}\) on a Banach space \(X\), with the growth estimate \(\|C(s)\| \leq M e^{\omega|s|}\) for \(s \in \mathbb{R}\). Let \(\nu \in M_\nu(\mathbb{R})\) and form the operator \(C_\nu\) as above. The idea of transference is to factorize the operator \(C_\nu\) over a convolution operator on a (vector-valued) function space. For (semi-)groups, the abstract principle was described recently in [17]. In the case of cosine functions the method is quite similar to the group case, due to the decoupling identity
\[
(5.12) \quad C(s+t) = C(s)C(t) - AS(s)S(t)
\]
Fourier multiplier norm of $\hat{\nu}$ where $\nu$ is convolution with the bounded measure $\mu$. This is the same as the Fourier multiplier norm in this case is just the $L_\infty$-norm, and so we arrive at an estimate

$$\|f(A)\| \lesssim \|f\|_{H^\infty(\Pi_\alpha)}$$

for $f \in F(\Pi_\alpha)$. By standard approximation methods (i.e., the “convergence lemma” mentioned in Section 3.2) this implies that $A$ has a bounded $H^\infty(\Pi_\alpha)$-calculus for each $\alpha > \omega$.

If $X$ is an UMD space then so is $X'$ and one arrives via the UMD-version of Mikhlīn’s theorem at other functional calculi. We do not go into detail here, since the result has already been described in [15, Thm. 5.3].

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