Characterization of exponential stability of a semigroup of operators in terms of its action by convolution on vector-valued function spaces over $\mathbb{R}_+$

J.M.A.M. van Neerven$^1$

Mathematisches Institut, Universität Tübingen
Auf der Morgenstelle 10, D-72076, Tübingen, Germany
E-mail: jane@michelangelo.mathematik.uni-tuebingen.de

It is proved that a $C_0$-semigroup $T = \{T(t)\}_{t \geq 0}$ of linear operators on a Banach space $X$ is uniformly exponentially stable if and only if it acts boundedly on one of the spaces $L^p(\mathbb{R}_+, X)$ or $C_0(\mathbb{R}_+, X)$ by convolution. As an application, it is shown that $T$ is uniformly exponentially stable if and only if

$$\sup_{s > 0} \left\| \int_0^s T(t)g(t) \, dt \right\| < \infty, \quad \forall g \in AP(\mathbb{R}_+, X),$$

where $AP(\mathbb{R}_+, X)$ is the space of $X$-valued almost periodic functions on $\mathbb{R}_+$.

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0. Introduction

It is well-known that the asymptotic behaviour of a $C_0$-semigroup of linear operators $T = \{T(t)\}_{t \geq 0}$ cannot be adequately described by the location of the spectrum $\sigma(A)$ of its infinitesimal generator $A$. If we define the growth bound $\omega_0(T)$ as the infimum of all $\omega \in \mathbb{R}$ for which a constant $M > 0$ exists such that

$$\|T(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0,$$

and the spectral bound $s(A)$ by

$$s(A) = \sup\{\text{Re}\lambda : \lambda \in \sigma(A)\},$$

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one has the inequality $s(A) \leq \omega_0(T)$, but strict inequality may occur. In their recent paper [LM], Latushkin and Montgomery-Smith proved the following striking fact: If one tensors an arbitrary $C_0$-semigroup $T$ on $X$ with the translation group on $L^p(\mathbb{R})$, the growth bound and the spectral bound of the resulting $C_0$-semigroup $S_p$ on $L^p(\mathbb{R}, X)$ coincide, and are equal to the growth bound of $T$. In fact, they proved the stronger result that the spectral mapping theorem holds for $S_p$. Independently, for $p = 2$ and $X$ Hilbert, this result was obtained by Rau [Ra1]. These results also hold for tensoring with the rotation group on $L^p()$, where $\Gamma$ is the unit circle. Furthermore, the analogous results hold for $C_0(\mathbb{R})$ and $C()$.

These results drew much attention and were extended into various directions; cf. [Ra2], [Ra3], [RS1], [Rs2], and the references given there. Also, L. Weis [W] recently announced a proof of the stability conjecture for positive $C_0$-semigroups on $L^p$ which is based on them.

The present paper was inspired by the Latushkin-Montgomery-Smith results in the following way. Explicitly, $S_p$ is given by

$$S_p(t)f(s) = T(t)f(s - t), \quad s \in \mathbb{R}, \quad t \geq 0, \quad f \in L^p(\mathbb{R}, X).$$

Integrating this over $[0, s]$ with respect to $t$, the resulting integral can be interpreted as the convolution of $T$ with the restriction $f|_{\mathbb{R}^+}$. Here, and in the rest of the paper, $\mathbb{R}^+ = [0, \infty)$. This observation motivates two questions: Firstly, do versions of the Latushkin-Montgomery-Smith theorem exist for the translation semigroups on $L^p(\mathbb{R}^+, X)$ and $C_0(\mathbb{R}^+, X)$, and secondly, can one study the asymptotic behaviour of $T$ by looking at its convolution with functions in $L^p(\mathbb{R}^+, X)$ or $C_0(\mathbb{R}^+, X)$? In this paper, we answer the second question affirmatively as follows. For a function $f \in L^1_{loc}(\mathbb{R}^+, X)$ we define the convolution $T \ast f$ by

$$(T \ast f)(s) := \int_0^s T(t)f(s - t)\, dt, \quad s \geq 0.$$

**Theorem 0.1.** Let $T$ be a $C_0$-semigroup on a Banach space $X$ and let $1 \leq p < \infty$. Then the following assertions are equivalent:

(i) $\omega_0(T) < 0$;

(ii) $T \ast f \in L^p(\mathbb{R}^+, X)$ for all $f \in L^p(\mathbb{R}^+, X)$;

(iii) $T \ast f \in C_0(\mathbb{R}^+, X)$ for all $f \in C_0(\mathbb{R}^+, X)$.

Loosely speaking, this means that $\omega_0(T) < 0$ if and only if $T$ acts on one of the spaces $L^p(\mathbb{R}^+, X)$ or $C_0(\mathbb{R}^+, X)$ by convolution. The proof, which is based on a partial answer to the first question, is given in Section 1. In Section 2, we apply the theorem to show that $\omega_0(T) < 0$ if and only if

$$\sup_{s > 0} \left\| \int_0^s T(t)f(t)\, dt \right\| < \infty, \quad \forall f \in AP(\mathbb{R}^+, X),$$

where $AP(\mathbb{R}^+, X)$ is the space of $X$-valued almost-periodic functions. In fact, we show that the same result is true for a more general class of spaces of $X$-valued functions.
which contains $C_0(\mathbb{R}_+, X)$ and $AP(\mathbb{R}_+, X)$. Also, we improve (iii) of Theorem 0.1 by showing that $\omega_0(T) < 0$ if $T * f \in L^\infty(\mathbb{R}_+, X)$ for all $f$ in one of the spaces in this class.

In Section 3, we discuss an extension of Theorem 0.1 to the action of $T$ by convolution on vector-valued Banach function spaces over $\mathbb{R}_+$.

In Section 4, we deal with the weak analogue of Theorem 0.1. It is shown that the condition that $(x^*, T * f) \in L^1(\mathbb{R}_+, X)$ for all $x^* \in X^*$ and $f \in L^1(\mathbb{R}_+, X)$ is equivalent to $\omega_0(T) < 0$ in Hilbert spaces. We also show that this need not be true in arbitrary Banach spaces.

1. Proof of Theorem 0.1

Let $T$ be a $C_0$-semigroup on a Banach space $X$. Let $1 \leq p < \infty$. We define a $C_0$-semigroup $S_p$ on $L^p(\mathbb{R}_+, X)$ by tensoring $T$ with the semigroup of right translations on $L^p(\mathbb{R}_+)$:

$$(S_p(t))f(s) = \begin{cases} T(t)f(s-t), & s-t \geq 0; \\ 0, & \text{else}. \end{cases}$$

We denote by $B_p$ its infinitesimal generator. Similarly, by using the right translation semigroup on $C_00(\mathbb{R}_+)$, the subspace of all $f \in C_0(\mathbb{R}_+)$ such that $f(0) = 0$, we obtain a $C_0$-semigroup $S_\infty$ on the corresponding $X$-valued space $C_00(\mathbb{R}_+, X)$. Its generator will be denoted by $B_\infty$.

For an operator $T$, we denote by $A\sigma(T)$ the approximate point spectrum of $T$.

**Lemma 1.1.** If $e^{i\theta} \in A\sigma(T(1))$ for some $\theta \in [0, 2\pi)$, then $0 \in A\sigma(B_p)$, $1 \leq p \leq \infty$.

**Proof:** The result is essentially contained in [RS2]. For reasons of self-containedness, and since the setting in [RS2] is more general and slightly different, we give the proof in some detail. First assume $1 \leq p < \infty$. Since $\theta \in A\sigma(T(1))$, for each $n = 1, 2, \ldots$ one can find a norm one vector $x_n \in X$ such that

$$\|T(k)x_n - e^{ik\theta}x_n\| \leq \frac{1}{2^k}, \quad k = 1, \ldots, n.$$ 

In particular, $\frac{1}{2} \leq \|T(k)x_n\| \leq 2$ for all $n$ and $k = 1, \ldots, n$. Using the local boundedness of $T$, it is easy to see that there are constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \leq \|T(t)x_n\| \leq \beta, \quad t \in [0, n]; \quad n = 1, 2, \ldots$$

For each $n$, let $a_n : \mathbb{R}_+ \to [0, 1]$ be a continuously differentiable function such that $a_n = 0$ on $[0, \frac{1}{8}] \cup [n - \frac{1}{8}, \infty)$, $a_n = 1$ on $[\frac{1}{8}, n - \frac{1}{8}]$, and $a_n'(t) \leq 10$ for all $t \geq 0$ and $n = 1, 2, \ldots$. Let $g_n := c_n^{-1}a_n(t)T(t)x_n$, where

$$c_n := \left( \int_0^n \|T(t)x_n\|^p dt \right)^{\frac{1}{p}}.$$
Then, using that \( a_n = 1 \) on \([\frac{1}{4}, n - \frac{1}{4}]\), we have

\[
\frac{\alpha}{2^\pi \beta} \leq \frac{\alpha(n - \frac{1}{2})^\frac{1}{p}}{\beta n^\frac{1}{p}} \leq \|g_n\|_{L^p(\mathbb{R}_+, X)} \leq 1; \quad n = 1, 2, \ldots
\]

Also, by direct calculation one checks that

\[
B_p g_n(t) = -c_n^{-1} a_n'(t) T(t) x_n, \quad t \geq 0.
\]

Since \(|a'_n| \leq 10\) on \([\frac{1}{8}, \frac{1}{4}] \cup [n - \frac{1}{4}, n - \frac{1}{8}]\) and 0 elsewhere, we have

\[
\|B_E g_n\|_{L^p(\mathbb{R}_+, X)} \leq \frac{10 \beta \cdot 8^{-\frac{1}{p}} + 10 \beta \cdot 8^{-\frac{1}{p}}}{\alpha n^\frac{1}{p}}; \quad n = 1, 2, \ldots
\]

Therefore \( \lim_{n \to \infty} \|B_p g_n\|_{L^p(\mathbb{R}_+, X)} = 0 \), which shows that \((\|g_n\|_{L^p(\mathbb{R}_+, X)}^{-1} \cdot g_n)_{n \geq 1}\) is an approximate eigenvector for \( B_p \) with approximate eigenvalue 0.

Next, we show how to modify this argument for \( p = \infty \). In this case, we choose a \( C^1 \)-function \( a_n \) that vanishes on \([0, \frac{1}{4}] \cup [n - \frac{1}{4}, \infty)\), and further satisfies \( a_n(\frac{1}{2} n) = 1 \), \( \|a_n\|_\infty = 1 \), and \( \|a'_n\|_\infty \leq 5n^{-1} \). Then \( g_n := a_n(t) T(t) x_n, n = 1, 2, \ldots, \) defines an approximate eigenvector for \( B_\infty \) with approximate eigenvalue 0. /////

**Lemma 1.12.** \( \text{With the above notations, for all } 1 \leq p \leq \infty \) we have \( s(B_p) = \omega_0(S_p) = \omega_0(T) \).

**Proof:** Clearly, \( s(B_p) \leq \omega_0(S_p) \leq \omega_0(T) \). It remains to prove that \( \omega_0(T) \leq s(B_p) \). By rescaling, we may assume that \( \omega_0(T) = 0 \), and it suffices to show that this implies that \( s(B_p) \geq 0 \). Since \( r(T(1)) = e^{\omega_0(T)} = 1 \), there is a \( \theta \in [0, 2\pi) \) such that \( e^{i\theta} \in \sigma(T(1)) \), and in fact, \( e^{i\theta} \in A \sigma(T(1)) \) since the boundary spectrum is always contained in the approximate point spectrum. By Lemma 1.1, this implies that \( 0 \in \sigma(B_p) \). Therefore, \( s(B_p) \geq 0 \) and the proof is complete. /////

**Theorem 1.3.** \( \text{Let } T \text{ be a } C_0 \text{-semigroup on a Banach space } X \text{ and let } 1 \leq p < \infty. \) Then the following assertions are equivalent:

(i) \( \omega_0(T) < 0 \);
(ii) \( T \ast f \in L^p(\mathbb{R}_+, X) \) for all \( f \in L^p(\mathbb{R}_+, X) \);
(iii) \( T \ast f \in C_0(\mathbb{R}_+, X) \) for all \( f \in C_0(\mathbb{R}_+, X) \).

**Proof:** First we prove (ii)\( \Rightarrow \) (i). By Lemma 1.2, we have to prove that \( s(B_p) < 0 \). For this, it is enough to prove that the resolvent of \( B_p \) exists and is uniformly bounded in the right half plane. Indeed, once this is established, a standard resolvent expansion argument shows that the resolvent exists and is uniformly bounded in a half plane \( \{\text{Re} z > -\epsilon\} \) for some \( \epsilon > 0 \); cf. [Hu] or [NSW]. Thus, \( s(B_p) \leq -\epsilon \).

We start by observing that there is a constant \( M > 0 \) such that \( \|T \ast f\|_{L^p(\mathbb{R}_+, X)} \leq M \|f\|_{L^p(\mathbb{R}_+, X)} \) for all \( f \in L^p(\mathbb{R}_+, X) \). To see this, we claim that the map \( f \mapsto T \ast f \) is closed as a map of \( L^p(\mathbb{R}_+, X) \) into itself. Indeed, assume \( f_n \to f \) and \( T \ast f_n \to g \) in
Then it is immediate that \((T \ast f_n)(s) \to (T \ast f)(s)\) for all \(s > 0\). On the other hand, since a norm convergent sequence in \(L^p(\mathbb{R}^+, X)\) contains a subsequence that converges pointwise a.e., it follows that \((T \ast f_{n_k})(s) \to g(s)\) for some sequence \((n_k)\) and almost all \(s\). Therefore, \(T \ast f = g\), as was to be shown. The existence of the constant \(M\) now follows from the closed graph theorem.

For \(T_0 > 0\), we define \(\pi_{T_0} : L^p(\mathbb{R}^+, X) \to L^p([0, T_0], X)\) by restriction: \(\pi_{T_0} f = f|_{[0,T_0]}\). For \(T > 0, T_0 > 0\) and \(f \in L^p(\mathbb{R}^+, X)\), we define the entire \(L^p(\mathbb{R}^+, X)\)-valued function \(F_{T,f}\) and the entire \(L^p([0, T_0], X)\)-valued function \(F_{T,T_0,f}\) by

\[
F_{T,f}(z) = \int_0^T e^{-zt} S_p(t) f(t) dt,

F_{T,T_0,f}(z) = \pi_{T_0}(F_{T,f}(z)).
\]

For each \(z\), the map \(f \mapsto F_{T,T_0,f}(z)\) is bounded as a map \(L^p(\mathbb{R}^+, X) \to L^p([0, T_0], X)\).

A trivial estimate shows that each of the functions \(z \mapsto F_{T,f}(z)\) and \(z \mapsto F_{T,T_0,f}(z)\) is bounded in each vertical strip \(\{0 < \text{Re} z < c\}, c > 0\).

For \(\lambda \in \mathbb{R}\) and \(f \in L^p(\mathbb{R}^+, X)\), let \(f_\lambda(s) := e^{i\lambda s} f(s), s \geq 0\). The restriction of \(S_p\) to the invariant subspace \(C_{00}(\mathbb{R}^+, X) \cap L^p(\mathbb{R}^+, X)\) extends to the \(C_0\)-semigroup \(S_{\infty}\) on \(C_{00}(\mathbb{R}^+, X)\). Since point evaluations on the latter space are continuous, for \(f \in C_{00}(\mathbb{R}^+, X) \cap L^p(\mathbb{R}^+, X), T \geq T_0, \text{ and } 0 \leq s \leq T_0\) we have

\[
\left( \int_0^T e^{-i\lambda t} S_p(t) f(t) dt \right)(s) = \int_0^T e^{-i\lambda t} S_p(t) f(s) dt = \int_0^s e^{-i\lambda t} T(t) f(s - t) dt = e^{-i\lambda s} \int_0^s T(t) f_\lambda(s - t) dt.
\]

Therefore, for \(T \geq T_0\),

\[
\| F_{T,T_0,f}(i\lambda) \|_{L^p([0,T_0], X)} = \| \pi_{T_0} \left( e^{-i\lambda (\cdot)} \int_0^{(\cdot)} T(t) f_\lambda(\cdot - t) dt \right) \|_{L^p([0,T_0], X)} \\
\leq \| \pi_{T_0} \| \cdot \| T \ast f_\lambda \|_{L^p(\mathbb{R}^+, X)} \leq \| T \ast f_\lambda \|_{L^p(\mathbb{R}^+, X)} \\
\leq M \| f_\lambda \|_{L^p(\mathbb{R}^+, X)} = M \| f \|_{L^p(\mathbb{R}^+, X)}.
\]

Since \(C_{00}(\mathbb{R}^+, X) \cap L^p(\mathbb{R}^+, X)\) is dense in \(L^p(\mathbb{R}^+, X)\), it follows that

\[
\| F_{T,T_0,f}(i\lambda) \|_{L^p([0,T_0], X)} \leq M \| f \|_{L^p(\mathbb{R}^+, X)}, \quad \forall \lambda \in \mathbb{R}, \quad T \geq T_0, \quad f \in L^p(\mathbb{R}^+, X).
\]

Also, if we choose constants \(N > 0\) and \(\omega \geq 0\) such that \(\| S_p(t) \| \leq Ne^{\omega t}\) for all \(t \geq 0\), then for \(\text{Re} z = \omega + 1\) we have

\[
\| F_{T,T_0,f}(z) \|_{L^p([0,T_0], X)} \leq \| \pi_{T_0} \| \int_0^T e^{-(\omega+1)t} Ne^{\omega t} \| f \|_{L^p(\mathbb{R}^+, X)} dt \\
\leq N(1 - e^{-T}) \| f \|_{L^p(\mathbb{R}^+, X)} \leq N \| f \|_{L^p(\mathbb{R}^+, X)}.
\]
It follows that for each \( f \in L^p(\mathbb{R}_+, X) \) and \( T_0 > 0 \) fixed, the functions \( z \mapsto F_{T,T_0,f}(z) \) are bounded on the line \( \text{Re} z = \omega + 1 \), uniformly with respect to \( T > 0 \), with bound \( N\|f\|_{L^p(\mathbb{R}_+, X)} \). Therefore, by (1.2) and the Phragmen-Lindelöf theorem [R, Thm. 12.8], for each \( f \) and \( T_0 \) fixed we have

\[
\|F_{T,T_0,f}(z)\|_{L^p([0,T_0],X)} \leq \max\{M,N\}\|f\|_{L^p(\mathbb{R}_+, X)}, \quad \forall 0 < \text{Re} z < \omega + 1, \quad T \geq T_0.
\tag{1.3}

Also, for \( \text{Re} z > \omega \) we have

\[
\lim_{T \to \infty} F_{T,T_0,f}(z) = \pi T_0 R(z, B_p)f.
\]

Combining these facts, it follows from Vitali’s theorem [HPh, Thm 3.14.1] that for each \( f \in L^p(\mathbb{R}_+, X) \) and \( T_0 > 0 \) the function \( z \mapsto \pi T_0 R(z, B_p)f \) has an analytic continuation \( F_{\infty,T_0,f} \) to \( \{0 < \text{Re} z < \omega + 1\} \), and that for \( 0 < \text{Re} z < \omega + 1 \),

\[
F_{\infty,T_0,f}(z) = \lim_{T \to \infty} F_{T,T_0,f}(z)
\]

uniformly on compacta. Moreover, by (1.3),

\[
\|F_{\infty,T_0,f}(z)\|_{L^p([0,T_0],X)} \leq \max\{M,N\}\|f\|_{L^p(\mathbb{R}_+, X)}, \quad \forall 0 < \text{Re} z < \omega + 1, \quad T_0 > 0.
\tag{1.4}

By regarding \( L^p([0,T_0], X) \) as a closed subspace of \( L^p(\mathbb{R}_+, X) \), for all \( \omega < \text{Re} z < \omega + 1 \) we have

\[
\lim_{T_0 \to \infty} F_{\infty,T_0,f}(z) = \lim_{T_0 \to \infty} \pi T_0 R(z, B_p)f = R(z, B_p)f,
\tag{1.5}

the convergence being with respect to the norm of \( L^p(\mathbb{R}_+, X) \). Again by Vitali’s theorem, now using (1.4), it follows that \( R(z, B_p)f \) admits a holomorphic extension \( F_{\infty,\infty,f} \) to \( \{0 < \text{Re} z < \omega + 1\} \), and that for all \( 0 < \text{Re} z < \omega + 1 \),

\[
\lim_{T_0 \to \infty} \lim_{T \to \infty} F_{T,T_0,f}(z) = \lim_{T_0 \to \infty} F_{\infty,T_0,f}(z) = F_{\infty,\infty,f}(z)
\tag{1.6}

uniformly on compacta. By an easy analytic continuation argument, we must have \( \{0 < \text{Re} z < \omega + 1\} \subset g(B_p) \) and \( F_{\infty,\infty,f}(z) = R(z, B_p)f \).

Therefore, by (1.4), (1.6), and the uniform boundedness theorem, it follows that \( R(z, B_p) \) is uniformly bounded in \( \{0 < \text{Re} z < \omega + 1\} \). By the Hille-Yosida theorem, \( R(z, B_p) \) is also uniformly bounded in \( \{\text{Re} z \geq \omega + 1\} \). Thus, \( R(z, B_p) \) exists and is uniformly bounded in \( \{\text{Re} z > 0\} \). This completes the proof of (ii) \( \Rightarrow \) (i).

Next, we prove (i) \( \Rightarrow \) (ii). Assume \( \omega_0(T) < 0 \), and choose \( \mu > 0 \) and \( M > 0 \) such that \( \|T(t)\| \leq Me^{-\mu t} \) for all \( t \geq 0 \). Let \( 1 \leq p < \infty \) be arbitrary and fixed. By applying Jensen’s inequality [R, Thm. 3.3] to the probability measure \( \mu(1 - e^{-\mu s})^{-1}e^{-\mu t} \, dt \) on
Since we note that there exists a constant \( C \) such that 
\[
\int_0^\infty \| \int_0^s T(t) f(s-t) dt \|^p ds \leq \int_0^\infty \left( \int_0^s M e^{-\mu t}\| f(s-t) \| dt \right)^p ds
\]
\[
\leq M^p \int_0^\infty \mu^{-p}(1 - e^{-\mu s})^p \int_0^s \| f(s-t) \|^p \mu(1 - e^{-\mu s})^{-1} e^{-\mu t} dt ds
\]
\[
\leq M^p \mu^{-(p-1)} \int_0^\infty \int_0^s e^{-\mu t} \| f(s-t) \|^p dt ds
\]
\[
= M^p \mu^{-(p-1)} \int_0^\infty e^{-\mu t} \int_t^\infty \| f(s-t) \|^p ds dt
\]
\[
\leq M^p \mu^{-p} \| f \|^p.
\]
This proves (i) \( \Rightarrow \) (ii).

The implication (i) \( \Rightarrow \) (iii) is proved as follows. Choose \( M > 0 \) and \( \mu > 0 \) such that 
\( \| T(t) \| \leq Me^{-\mu t} \) for all \( t \geq 0 \). Fix \( \epsilon > 0 \) and \( f \in C_0(\mathbb{R}_+, X) \) arbitrary. Choose \( N \) so large that \( se^{-\mu s} \leq \epsilon \) and 
\( \| f(s) \| \leq \| f \|_{C_0(\mathbb{R}_+, X)} \) for all \( s \geq N \). Then, for \( s \geq 2N \),
\[
\left\| \int_0^s T(t) f(s-t) dt \right\| \leq \int_0^s M e^{-\mu t} \| f \|_{C_0(\mathbb{R}_+, X)} dt + \int_{s}^{s+1} M e^{-\mu t} \| f \|_{C_0(\mathbb{R}_+, X)} dt
\]
\[
\leq M(1 + \mu^{-1}) \epsilon \| f \|_{C_0(\mathbb{R}_+, X)}.
\]
Since \( \mathbf{T} \ast f \) also is continuous, we obtain the desired conclusion.

It remains to prove (iii) \( \Rightarrow \) (i). We do this by modifying the proof of (ii) \( \Rightarrow \) (i). First we note that there exists a constant \( M > 0 \) such that
\[
\| \mathbf{T} \ast f \|_{C_0(\mathbb{R}_+, X)} \leq M \| f \|_{C_0(\mathbb{R}_+, X)} \quad \forall f \in C_0(\mathbb{R}_+, X).
\]
Indeed, this follows from applying the uniform boundedness theorem to the operators 
\( T_s : f \mapsto \int_0^s T(t) f(s-t) dt \).

Let \( f \in C_{00}(\mathbb{R}_+, X) \) arbitrary. Since \( \mathbf{T} \ast f \in C_0(\mathbb{R}_+, X) \) by assumption and 
\( (\mathbf{T} \ast f)(0) = 0 \), it follows that \( \mathbf{T} \) acts boundedly on \( C_{00}(\mathbb{R}_+, X) \) by convolution, with norm at most \( M \).

Let \( C_0([0, T_0], X) \) be the closed subspace of \( C_{00}(\mathbb{R}_+, X) \) consisting of all functions vanishing on \([T_0, \infty)\). For each \( T_0 \geq 1 \), define the piecewise linear function \( g_{T_0} \) on \( \mathbb{R}_+ \) by
\[
g_{T_0}(t) = \begin{cases} 1, & 0 \leq t \leq T_0 - 1; \\ T_0 - t, & T_0 - 1 \leq t \leq T_0; \\ 0, & \text{else} \end{cases}
\]
and the operator \( \Pi_{T_0} : C_{00}(\mathbb{R}_+, X) \to C_{00}([0, T_0], X) \) by
\[
(\Pi_{T_0} f)(t) = g_{T_0}(t) f(t), \quad 0 \leq t \leq T_0.
\]
Note that, for any function \( f \in C_{00}(\mathbb{R}_+, X) \), 
\( \| \Pi_{T_0} f \|_{C_{00}([0, T_0], X)} \leq \| f \|_{C_{00}(\mathbb{R}_+, X)} \).

With \( \pi_{T_0} \) replaced by \( \Pi_{T_0} \), the proof of now proceeds along the lines of (ii) \( \Rightarrow \) (i).
The reason for introducing the operators $\Pi T_0$ is as follows. If we simply truncate a function in $C_{00}(\mathbb{R}_+, X)$ with $\pi T_0$, the resulting function need not define an element of $C_{00}([0, T_0], X)$, so that we cannot perform the limiting operation (1.5). With the operator $\Pi T_0$, this poses no problems.

In the next section, we will improve part of Theorem 1.3 by showing that $\omega_0(T) < 0$ if (and only if) $T * f$ is merely bounded for all $f \in C_0(\mathbb{R}_+, X)$.

2. Applications

The main result of this section is an application of Theorem 1.3 for the case of $C_0(\mathbb{R}_+, X)$. We need the following definitions in order to state the result. Let $BUC(\mathbb{R}_+, X)$ denote the space of all $X$-valued bounded uniformly continuous functions on $\mathbb{R}_+$. A linear subspace $E$ of $BUC(\mathbb{R}_+, X)$ will be called locally dense in $BUC(\mathbb{R}_+, X)$ if for every $I \in \mathbb{R}_+$, and every $f \in C(I)$ there exists a function $f_{\epsilon,I} \in E$ such that

$\sup_{t \in I} \| f(t) - f_{\epsilon,I}(t) \| \leq \epsilon.$

If, in addition, there is a constant $K > 0$, independent of $I$ and $\epsilon$, such that for every $f \in C(I)$ the function $f_{\epsilon,I}$ can be chosen in such a way that $\| f_{\epsilon,I} \|_{BUC(\mathbb{R}_+, X)} \leq K \| f \|_{C(I)}$, we say that $E$ is boundedly locally dense in $BUC(\mathbb{R}_+, X)$.

**Theorem 2.1.** Let $T$ be a $C_0$-semigroup a Banach space $X$ and let $E$ be a closed, boundedly locally dense subspace of $BUC(\mathbb{R}_+, X)$. Then the following assertions are equivalent:

(i) $\omega_0(T) < 0$;

(ii) $\sup_{s > 0} \left\| \int_0^s T(t)g(t)\, dt \right\| < \infty$ for all $g \in E$.

**Proof:** The implication (i) $\Rightarrow$ (ii) is trivial. We will prove (ii) $\Rightarrow$ (i). By the uniform boundedness theorem, there is a constant $C > 0$ such that

$$\sup_{s > 0} \left\| \int_0^s T(t)g(t)\, dt \right\| \leq C\| g \|_{BUC(\mathbb{R}_+, X)}, \quad \forall g \in E. \quad (2.1)$$

For a given $f \in C_0(\mathbb{R}_+, X)$ and $s > 0$, let $M_s = \sup_{0 \leq t \leq s} \| T(t) \|$ and let $f_s \in E$ be any function such that

$$\sup_{0 \leq t \leq s} \| f(s - t) - f_s(t) \| \leq \frac{1}{s M_s} \| f \|_{C_0(\mathbb{R}_+, X)},$$

and

$$\| f_s \|_{BUC(\mathbb{R}_+, X)} \leq K\| f \|_{C_0(\mathbb{R}_+, X)}.$$
Such an $f_s$ exists by the definition of a boundedly locally dense subspace; $K$ is the constant from the definition. Then, by (2.1),

$$
\| (T * f)(s) \| = \left\| \int_0^s T(t) (f(s-t) - f_s(t) + f_s(t)) \, dt \right\| \\
\leq \| f \|_{C_0(\mathbb{R}_+, X)} + \left\| \int_0^s T(t)f_s(t) \, dt \right\| \\
\leq \| f \|_{C_0(\mathbb{R}_+, X)} + C\| f_s \|_{BUC(\mathbb{R}_+, X)} \leq (1 + CK)\| f \|_{C_0(\mathbb{R}_+, X)}.
$$

(2.2)

It follows that $T * f$ is a bounded continuous function. If we can prove that

$$
\lim_{s \to \infty} \int_0^s T(t)f(s-t) \, dt = 0,
$$

it follows that convolution with $T$ maps $C_0(\mathbb{R}_+, X)$ into $C_0(\mathbb{R}_+, X)$. Then we can apply Theorem 1.3 to obtain that $\omega_0(T) < 0$.

Fix $\varepsilon > 0$ arbitrary and choose $N$ so large that $\| f(s) \| \leq (1 + CK)^{-1}\varepsilon$ for all $s \geq N$. Write $f = f_0 + f_1$, where $f_0 \in C_0(\mathbb{R}_+, X)$ is chosen in such a way that $f_0(s) = f(s)$ for all $s \geq N$ and $\| f_0 \|_{C_0(\mathbb{R}_+, X)} \leq (1 + CK)^{-1}\varepsilon$. Note that the support of $f_1$ is contained in the interval $[0, N]$. By (2.2), for all $s \geq 0$ we have

$$
\left\| \int_0^s T(t)f_0(s-t) \, dt \right\| \leq (1 + CK)\| f_0 \|_{C_0(\mathbb{R}_+, X)} \leq \varepsilon.
$$

It follows that it is sufficient to prove that

$$
\lim_{s \to \infty} T(s-N) \left( \int_0^N T(t)f_1(N-t) \, dt \right) = \lim_{s \to \infty} \int_0^s T(t)f_1(s-t) \, dt = 0.
$$

(2.3)

Since $\lim_{\lambda \to \infty} \lambda^2 R(\lambda, A)^2 f_1(\cdot) \to f_1(\cdot)$ uniformly on $[0, N]$, and hence on all of $\mathbb{R}_+$, (2.2) shows that it is even sufficient to prove that

$$
\lim_{s \to \infty} T(s-N) \left( \int_0^N T(t)\lambda^2 R(\lambda, A)^2 f_1(N-t) \, dt \right) = 0
$$

(2.4)

for all $\lambda$ sufficiently large. Note that, for each such $\lambda$,

$$
\int_0^N T(t)\lambda^2 R(\lambda, A)^2 f_1(N-t) \, dt \in D(A^2).
$$

(2.5)

In order to prove (2.4), we claim that the resolvent of $A$ exists and is uniformly bounded in the right halfplane.

To prove this, fix $\mu > 0$, $g \in E$, and $s > 0$ arbitrary. Choose a function $g \in E$ such that

$$
\sup_{0 \leq t \leq s} \| g_{\mu,s}(t) - e^{-\mu t}g(t) \| \leq \frac{1}{\mu M_s} \| g \|_{BUC(\mathbb{R}_+, X)}
$$

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and \( \|g_{\mu,s}\|_{\text{BUC}(\mathbb{R}_+,X)} \leq K\|g\|_{\text{BUC}(\mathbb{R}_+,X)} \). Then, by (2.1),
\[
\left\| \int_0^s e^{-\mu t} T(t) g(t) \, dt \right\| \leq \|g\|_{\text{BUC}(\mathbb{R}_+,X)} + C\|g_{\mu,s}\|_{\text{BUC}(\mathbb{R}_+,X)}
\leq (1 + CK)\|g\|_{\text{BUC}(\mathbb{R}_+,X)}.
\] (2.6)

Next, let \( \nu \in \mathbb{R} \) and \( x \in X \) be arbitrary and fixed. Let \( g_{\nu,x,s} \in E \) be a function such that
\[
\sup_{0 \leq t \leq s} \|g_{\nu,x,s}(t) - e^{-i\nu t} \otimes x\| \leq \frac{1}{sM_s}\|x\|
\]
and \( \|g_{\nu,x,s}\|_{\text{BUC}(\mathbb{R}_+,X)} \leq K\|x\| \). Then, by (2.6) applied to \( g_{\nu,x,s} \),
\[
\left\| \int_0^s e^{-(\mu+i\nu)t} T(t)x \, dt \right\| \leq \|x\| + \left\| \int_0^s e^{-\mu t} T(t)g_{\nu,x,s}(t) \, dt \right\|
\leq \|x\| + (1 + CK)\|g_{\nu,x,s}\|_{\text{BUC}(\mathbb{R}_+,X)}
\leq (1 + (1 + CK)K)\|x\|.
\] (2.7)

Since \( s > 0 \) and \( \mu > 0, \nu > 0 \) are arbitrary, (2.7) shows that the entire \( X \)-valued functions \( z \mapsto \int_0^s e^{-zt} T(t)x \, dt \) are bounded in the right half plane, uniformly in \( s > 0 \). Moreover, for \( \text{Re} \, z \) sufficiently large, they converge to \( R(z,A)x \) as \( s \to \infty \). An application of Vitali’s theorem and an analytic continuation argument show that the right half plane is contained in \( g(A) \) and that the resolvent of \( A \) is uniformly bounded there. This concludes the proof of the claim.

By a well-known theorem of Slemrod [Sl], the uniform boundedness of the resolvent in the right half plane implies that \( \omega_2(T) < 0 \), i.e. there are constants \( K > 0 \) and \( \mu > 0 \) such that \( \|T(t)y\| \leq Ke^{-\mu t}\|y\|_{D(A^2)} \) for all \( y \in D(A^2) \) and \( t \geq 0 \). Therefore, (2.4) is a consequence of (2.5).

Since \( C_0(\mathbb{R}_+,X) \) is a closed, boundedly locally dense subspace of \( \text{BUC}(\mathbb{R}_+,X) \), Theorem 2.1 implies:

**Corollary 2.2.** Let \( T \) be a \( C_0 \)-semigroup on a Banach space \( X \). Then \( \omega_0(T) < 0 \) if and only if
\[
\sup_{s > 0} \left\| \int_0^s T(t)g(t) \, dt \right\| < \infty, \quad \forall g \in C_0(\mathbb{R}_+,X).
\]

As another application, we define \( AP(\mathbb{R}_+,X) \) as the closure in \( \text{BUC}(\mathbb{R}_+,X) \) of the set of all functions \( \{e^{i\lambda t} \otimes x : \lambda \in \mathbb{R}, x \in X \} \). It is easy to see that \( AP(\mathbb{R}_+,X) \) is boundedly locally dense in \( \text{BUC}(\mathbb{R}_+,X) \). Indeed, if \( I \subset \mathbb{R}_+ \) is a bounded closed interval and \( f \in C(I) \) is given, we choose \( N \) so large that \( I \subset [0,N] \) and fix an arbitrary continuous function \( f_N \in C([0,N+1]) \) that coincides with \( f \) on \( I \) and satisfies \( f(0) = f(N+1) \). Then we approximate \( f_N \) uniformly in \( [0,N+1] \) by linear combinations of functions \( e^{i\delta t} \otimes x, \delta \in \{2\pi k/(N+1) : k \in \mathbb{Z}\}, x \in X \). Since these functions are \( N+1 \)-periodic, their sup-norms on \( \mathbb{R}_+ \) are the same as their sup-norms in \( [0,N+1] \). Therefore, \( AP(\mathbb{R}_+,X) \) is boundedly locally dense in \( \text{BUC}(\mathbb{R}_+,X) \). Since \( AP(\mathbb{R}_+,X) \) is also closed in \( \text{BUC}(\mathbb{R}_+,X) \), we obtain:
Corollary 2.3. Let $T$ be a $C_0$-semigroup on a Banach space $X$. Then $\omega_0(T) < 0$ if and only if
\[
\sup_{s > 0} \left\| \int_0^s T(t)g(t) \, dt \right\| < \infty, \quad \forall g \in AP(\mathbb{R}+, X). \tag{2.8}
\]

One should compare this theorem to the following result of [Ne]: if
\[
\sup_{\lambda \in \mathbb{R}} \sup_{s > 0} \left\| \int_0^s e^{i\lambda t}T(t)x \, dt \right\| < \infty, \tag{2.9}
\]
then $\omega_1(T) < 0$. Here, $\omega_1(T)$ denotes the growth bound of orbits originating from $D(A)$, i.e. the infimum of all $\omega \in \mathbb{R}$ for which there is an $M > 0$ such that $\|T(t)x\| \leq Me^{\omega t}\|x\|_{D(A)}$ for all $x \in D(A)$. The difference between (2.8) and (2.9) is that in the latter we consider one function $e^{i\lambda(t)} \otimes x \in AP(\mathbb{R}+, X)$ at a time, whereas in (2.8) we consider all functions in $AP(\mathbb{R}+, X)$. The supremum over $\lambda \in \mathbb{R}$ in (2.9) accounts for the fact that the sup-norms of $e^{i\lambda(t)} \otimes x$ are uniform in $\lambda$.

We conclude this section with an improvement of Theorem 1.3 for the case $C_0(\mathbb{R}+, X)$.

Theorem 2.4. Let $T$ be a $C_0$-semigroup a Banach space $X$ and let $E$ be a closed, boundedly locally dense subspace of $BUC(\mathbb{R}+, X)$. Then the following assertions are equivalent:

(i) $\omega_0(T) < 0$;

(ii) $T \ast f \in L^\infty(\mathbb{R}+, X)$ for all $f \in E$.

Proof: The implication (i)$\Rightarrow$(ii) is trivial. We will prove (ii)$\Rightarrow$(i). By the uniform boundedness theorem applied to the operators $T_s : f \mapsto \int_0^s T(t)f(s-t) \, dt$, there is a constant $C > 0$ such that
\[
\sup_{s > 0} \left\| \int_0^s T(t)f(s-t) \, dt \right\| \leq C\|f\|_{BUC(\mathbb{R}+, X)}, \quad \forall f \in E. \tag{2.10}
\]

For a given $f \in E$ and $s > 0$, let $M_s = \sup_{0 \leq t \leq s} \|T(t)\|$ and let $f_s \in E$ be any function such that
\[
\sup_{0 \leq t \leq s} \|f(s-t) - f_s(t)\| \leq \frac{1}{sM_s}\|f\|_{BUC(\mathbb{R}+, X)},
\]
and
\[
\|f_s\|_{BUC(\mathbb{R}+, X)} \leq K\|f\|_{BUC(\mathbb{R}+, X)}.
\]

Then, by (2.10),
\[
\left\| \int_0^s T(t)f(t) \, dt \right\| \leq \|f\|_{BUC(\mathbb{R}+, X)} + \left\| \int_0^s T(t)f_s(s-t) \, dt \right\|
\leq \|f\|_{BUC(\mathbb{R}+, X)} + C\|f_s\|_{BUC(\mathbb{R}+, X)} \leq (1 + CK)\|f\|_{BUC(\mathbb{R}+, X)}.
\]

Since $s > 0$ is arbitrary, $\omega_0(T) < 0$ by Theorem 2.1. //
3. Generalization to Banach function spaces

Let $E$ be a rearrangement invariant Banach function space over $\mathbb{R}_+$. We will adopt the terminology of the book [BS]. Although the definition of a Banach function space given there is very restrictive, it is not difficult to show that in the rearrangement invariant case the assumptions (P4) and (P5) of [BS], Def. I.1.1, are redundant provided one assumes that $E$ is carried by $\mathbb{R}_+$. This means that for each measurable $H \subset \mathbb{R}_+$ of positive measure there exists a function $f \in E$ that is not identically zero a.e. on $H$.

The fundamental function is the function $\varphi_E$ defined by

$$\varphi_E(t) := \|\chi_H\|_E,$$

where $H$ is a subset of measure $t$. One can show that for every $t \geq 0$, such a set exists. By the rearrangement invariance, the function $\varphi_E$ is well-defined.

By [BS, Lemma III.6.3], the right translation semigroup is strongly continuous on $E$ if and only if $\varphi_E(0+) = 0$, provided the simple functions are dense in $E$. This is the case if $E$ has order continuous norm, which in turn is the case if $E$ is separable (this follows from [BS, Chapters I.4 and I.5]). Examples of rearrangement invariant Banach function spaces with order continuous norm satisfying $\varphi_E(0+) = 0$ are the spaces $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$, and all reflexive spaces rearrangement invariant Banach function spaces.

If $E = E(\mathbb{R}_+)$ is a rearrangement invariant Banach function space over $\mathbb{R}_+$ with order continuous norm satisfying $\varphi_E(0+) = 0$, the operators $S_E(t)$ defined by

$$S_E(t)f(s) := \begin{cases} T(t)f(s-t), & s-t \geq 0; \\ 0, & \text{else,} \end{cases}$$

define a $C_0$-semigroup $S_E$ on $E(\mathbb{R}_+,X)$. It generator will be denoted by $B_E$. Here, $E(\mathbb{R}_+,X)$ is the space of all Bochner measurable functions $f : \mathbb{R}_+ \to X$ such that $\|f(\cdot)\| \in E(\mathbb{R}_+)$. This space is a Banach space, as can be seen by a modification of the proof of the completeness of $L^p$.

It is not hard to see that the proofs of Lemmas 1.1 and 1.2 carry over almost verbatim to $S_E$. One has to distinguish between the cases that $\lim_{t \to \infty} \varphi_E(t)$ is infinite or finite. In the first case, one argues as for $L^p(\mathbb{R}_+)$ and in the second case as for $C_0(\mathbb{R}_+)$. Summarizing, we have:

**Lemma 3.1.** Let $T$ be a $C_0$-semigroup on $X$ and let $E(\mathbb{R}_+)$ be a rearrangement invariant Banach function space with order continuous norm satisfying $\varphi_E(0+) = 0$. Then, for the semigroup $S_E$ we have $s(B_E) = \omega_0(S_E) = \omega_0(T)$.

**Theorem 3.2.** Let $T$ be a $C_0$-semigroup on $X$ and let $E(\mathbb{R}_+)$ be a rearrangement invariant Banach function space with order continuous norm satisfying $\varphi_E(0+) = 0$. Let $T$ be a $C_0$-semigroup on a Banach space $X$. Then the following assertions are equivalent:

(i) $\omega_0(T) < 0$;

(ii) $T * f \in E(\mathbb{R}_+,X)$ for all $f \in E(\mathbb{R}_+,X)$. 

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Proof: (ii)⇒(i): Armed with Lemma 3.1, we can copy the proof of (ii)⇒(i) of Theorem 1.3 almost verbatim. In order to establish the boundedness of convolution with \( T \) as a map of \( E(\mathbb{R}_+, X) \) to itself, we now use the fact that a norm convergent sequence in \( E(\mathbb{R}_+, X) \) has a subsequence that converges pointwise a.e. This in turn follows from the fact that \( f_n \to f \) in \( E(\mathbb{R}_+, X) \) implies that \( f_n|_{[0,k]} \to f|_{[0,k]} \) in the norm of \( L^1[0,k] \) for all \( k = 1, 2, ... \) [BS, Cor. 11.2] and a diagonal argument.

The proof that (i) implies (ii) proceeds as follows. For \( \mu > 0 \) and \( g \in L^1_{loc}(\mathbb{R}_+) \), define

\[
(T_\mu(g))(s) := \int_0^s e^{-\mu t} g(s - t) \, dt, \quad s \geq 0.
\]

The proof of Theorem 1.3, (i)⇒(ii), shows that this defines a bounded operator \( T_\mu : L^1(\mathbb{R}_+) \to L^1(\mathbb{R}_+) \) of norm \( \leq \mu^{-1} \). Also, it is trivial that \( T_\mu \) is bounded as an operator \( L^\infty(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+) \), of norm \( \leq \mu^{-1} \). By a well-known theorem of Calderón [C] (see also [BS, Thm. III.2.12]), every rearrangement invariant Banach function space over \( \mathbb{R}_+ \) is an exact interpolation space between \( L^1(\mathbb{R}_+) \) and \( L^\infty(\mathbb{R}_+) \). Therefore, \( T_\mu \) is bounded as an operator \( E(\mathbb{R}_+) \to E(\mathbb{R}_+) \), of norm \( \leq \mu^{-1} \).

Let \( f \in E(\mathbb{R}_+, X) \) be arbitrary. Since \( \omega_0(T) < 0 \), there are constants \( M > 0 \) and \( \mu > 0 \) such that \( \|T(t)\| \leq Me^{-\mu t} \) for all \( t \geq 0 \). Since \( \|f(\cdot)\| \in E(\mathbb{R}_+) \), we have

\[
\|T * f\|_{E(\mathbb{R}_+, X)} = \left\| \left( T * f \right)(\cdot) \right\|_{E(\mathbb{R}_+)} = \left\| \int_0^\infty Me^{-\mu t} \|f(\cdot - t)\| \, dt \right\|_{E(\mathbb{R}_+)} \leq M \left\| f \right\|_{E(\mathbb{R}_+)}.
\]

//\

4. The weak case

In this section, we study the weak analogue of Theorem 1.3. If \( E(\mathbb{R}_+) \) is a given function space over \( \mathbb{R}_+ \), we want to characterize those semigroups \( T \) on \( X \) for which \( \langle x^*, T * f \rangle \) defines an element of \( E(\mathbb{R}_+) \) for all \( x^* \in X^* \) and \( f \in E(\mathbb{R}_+, X) \). Here, and in the following, for a \( g \in L^1_{loc}(\mathbb{R}_+, X) \) and a functional \( x^* \in X^* \), the function \( \langle x^*, g \rangle \in L^1_{loc}(\mathbb{R}_+) \) is defined in the natural way: \( \langle x^*, g \rangle(s) = \langle x^*, g(s) \rangle; s \geq 0. \)

For \( E = L^1 \) we solve this problem as follows. A semigroup \( T \) is said to be weakly \( L^1 \)

\[
\int_0^\infty |\langle x^*, T(t)x \rangle| \, dt < \infty, \quad \forall x \in X, x^* \in X^*.
\]

**Theorem 4.1.** Let \( T \) be a \( C_0 \)-semigroup on a Banach space \( X \). Then the following assertions are equivalent:

(i) \( T \) is weakly \( L^1 \);
(ii) \( \langle x^*, T * f \rangle \in L^1(\mathbb{R}_+) \) for all \( f \in L^1(\mathbb{R}_+, X) \) and \( x^* \in X^* \).
Proof: Assume (ii). Let $S : Y_0 \times Y_1 \to Z$ be a separately continuous bilinear map. For $y_0 \in Y_0$ define $S_0 : Y_1 \to Z$, $S_{y_0}(y_1) := S(y_0, y_1)$. Then each $S_{y_0}$ is bounded by the continuity in the $Y_1$-variable. Using the continuity in the $Y_0$-variable, it is easy to see that the map $y_0 \mapsto S_{y_0}$ is closed, and hence bounded by the closed graph theorem. It follows that there is an $M > 0$ such that

$$\|S(y_0, y_1)\| \leq \|S_{y_0}\| \|y_1\| \leq M \|y_0\| \|y_1\|.$$ 

Applying this to the separately continuous bilinear map $T : X^* \times L^1(\mathbb{R}_+, X) \to L^1(\mathbb{R}_+)$ defined by $T(x^*, f) = \langle x^*, T * f \rangle$, it follows that there exists an $M > 0$ such that

$$\|(x^*, T * f)\|_{L^1(\mathbb{R}_+)} \leq M \|f\|_{L^1(\mathbb{R}_+, X)} \|x^*\|, \quad \forall f \in L^1(\mathbb{R}_+, X), x^* \in X^*.$$ 

Fix $x \in X$, $x^* \in X^*$ and $s_0 > 1$ arbitrary. Choose $0 < s_0 < 1$ such that

$$\frac{1}{s_0} \left| \int_0^{s_0} \langle x^*, T(s-t)x \rangle \, dt \right| \geq \frac{1}{2} \left| \langle x^*, T(s)x \rangle \right|, \quad \forall 1 \leq s \leq s_0.$$ 

Then,

$$\int_1^{s_0} \left| \langle x^*, T(s)x \rangle \right| \, ds \leq 2 \int_1^{s_0} \frac{1}{s_0} \left| \int_0^{s_0} \langle x^*, T(s-t)x \rangle \, dt \right| \, ds$$

$$= 2 \int_1^{s_0} \frac{1}{s_0} \left| \int_0^s \langle x^*, T(t)x \rangle \chi_{[0,s]}(s-t) \, dt \right| \, ds$$

$$\leq \frac{2}{s_0} \|(x^*, T * (\chi_{[0,s_0]} \otimes x))\|_{L^1(\mathbb{R}_+)}$$

$$\leq \frac{2M}{s_0} \|\chi_{[0,s_0]} \otimes x\|_{L^1(\mathbb{R}_+, X)} \|x^*\| = 2M \|x\| \|x^*\|.$$ 

Since $s_0 > 1$ is arbitrary, it follows that

$$\int_1^{s_0} \left| \langle x^*, T(s)x \rangle \right| \, ds \leq 2M \|x\| \|x^*\|, \quad \forall x \in X, x^* \in X^*.$$ 

Therefore, $\int_0^{\infty} \left| \langle x^*, T(s)x \rangle \right| \, ds < \infty$ for all $x \in X$ and $x^* \in X^*$, which proves (i).

Now assume (i). As is well-known and easy to see, there exists a constant $C$ such that

$$\int_0^{\infty} \left| \langle x^*, T(t)x \rangle \right| \, dt \leq C \|x\| \|x^*\|, \quad \forall x \in X, x^* \in X^*.$$ 

Let $N := \sup_{0 \leq s \leq 1} \|T(s)\|$. Fix $x \in X$, $x^* \in X^*$ and real numbers $0 \leq t_0 < t_1$.
with \( t_1 - t_0 \leq 1 \). Then,
\[
\int_0^\infty \left| \left\langle x^*, \int_0^s T(\tau)(\chi_{[t_0,t_1]} \otimes x)(s-\tau) \, d\tau \right\rangle \right| \, ds
\]
\[
= \int_0^\infty \left| \left\langle x^*, \int_{\max\{s-t_0,0\}}^{\max\{s-t_1,0\}} T(\tau) x \, d\tau \right\rangle \right| \, ds
\]
\[
\leq \int_{t_0}^{t_1} \left| \left\langle x^*, \int_0^{s-t_0} T(\tau) x \, d\tau \right\rangle \right| \, ds + \int_{t_1}^\infty \left| \left\langle x^*, \int_{s-t_1}^{s-t_0} T(\tau) x \, d\tau \right\rangle \right| \, ds
\]
\[
\leq \int_{t_0}^{t_1} (s-t_0) N \| x \| \| x^* \| \, ds + \int_0^\infty \left| \left\langle x^*, T(s) \int_0^{t_1-t_0} T(\tau) x \, d\tau \right\rangle \right| \, ds
\]
\[
\leq (t_1 - t_0) N \| x \| \| x^* \| + C \left\| \int_0^{t_1-t_0} T(\tau) x \, d\tau \right\| \| x^* \|
\]
\[
\leq (t_1 - t_0) N (1 + C) \| x \| \| x^* \|.
\]

Therefore, with \( M = N(1 + C) \), we have
\[
\| \left\langle x^*, T \ast (\chi_{[t_0,t_1]} \otimes x) \right\rangle \|_{L^1(\mathbb{R})} \leq M (t_1 - t_0) \| x \| \| x^* \|
\]

Next, let \( f \) be a stepfunction of the form \( f = \sum_{k=0}^{n-1} \chi_{[t_k,t_{k+1}] \otimes x_k} \). By splitting large intervals into finitely many smaller ones, we may assume that \( 0 < t_{k+1} - t_k \leq 1 \) for all \( k = 0, \ldots, n-1 \). By the above estimate we have
\[
\| \left\langle x^*, T \ast f \right\rangle \|_{L^1(\mathbb{R})} \leq \sum_{k=0}^{n-1} \| \left\langle x^*, T \ast (\chi_{[t_k,t_{k+1}] \otimes x_k}) \right\rangle \|_{L^1(\mathbb{R})}
\]
\[
\leq M \sum_{k=0}^{n-1} (t_{k+1} - t_k) \| x_k \| \| x^* \| = M \| f \|_{L^1(\mathbb{R},X)} \| x^* \|.
\]

Since the stepfunctions supported by finite unions of intervals of length \( \leq 1 \) are dense in \( L^1(\mathbb{R},X) \), (4.1) holds for arbitrary \( f \in L^1(\mathbb{R},X) \). This proves the implication \((i) \Rightarrow (ii)\).

Since there exist weakly \( L^1 \) semigroups with positive growth bound, the theorem shows that condition (ii) does not characterize exponential stability. In Hilbert space however, a \( C_0 \)-semigroup \( T \) is weakly \( L^1 \) if and only if \( \omega_0(T) < 0 \) [HK], [We]. In combination with Theorem 4.1, this leads to the following result.

**Corollary 4.2.** Let \( T \) be a \( C_0 \)-semigroup on a Hilbert space \( H \). Then the following assertions are equivalent:

(i) \( \omega_0(T) < 0 \);

(ii) \( \langle y, T \ast f \rangle \in L^1(\mathbb{R}) \) for all \( f \in L^1(\mathbb{R},H) \) and \( y \in H \).

A positive \( C_0 \)-semigroup on a Banach lattice \( X \) is weakly \( L^1 \) if and only if \( s(A) < 0 \). This is more or less folklore; a proof can be found in [NSW].

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Corollary 4.3. Let $T$ be a positive $C_0$-semigroup with generator $A$ on a Banach lattice $X$. Then the following assertions are equivalent:

(i) $s(A) < 0$;
(ii) $(x^*, T \ast f) \in L^1(\mathbb{R}_+)$ for all $f \in L^1(\mathbb{R}_+, X)$ and $x^* \in X$.

5. References


