Let $H$ be a Hilbert space and $E$ a Banach space. In this note we present a sufficient condition for an operator $R: H \to E$ to be $\gamma$–radonifying in terms of Riesz sequences in $H$. This result is applied to recover a result of Lutz Weis and the second named author on the $R$-boundedness of resolvents, which is used to obtain a Datko-Pazy type theorem for the stochastic Cauchy problem. We also present some perturbation results.

1 Introduction

The well-known Datko-Pazy theorem states that if $(T(t))_{t \geq 0}$ is a strongly continuous semigroup on a Banach space $E$ such that all orbits $T(\cdot)x$ belong to the space $L^p(\mathbb{R}_+, E)$ for some $p \in [1, \infty)$, then $(T(t))_{t \geq 0}$ is uniformly exponentially stable, or equivalently, there exists an $\varepsilon > 0$ such that all orbits $t \mapsto e^{\varepsilon t}T(t)x$ belong to $L^p(\mathbb{R}_+, E)$. For $p = 2$ and Hilbert spaces $E$ this result is due to Datko [3], and the general case was obtained by Pazy [11].

In this note we prove a stochastic version of the Datko-Pazy theorem for spaces of $\gamma$–radonifying operators (cf. Section 2). Let us denote by $\gamma(\mathbb{R}_+, E)$ the space of all strongly measurable functions $\phi: \mathbb{R}_+ \to E$ for which the integral operator

$$f \mapsto \int_0^\infty f(t)\phi(t)\, dt$$

is well-defined and $\gamma$-radonifying from $L^2(\mathbb{R}_+)$ to $E$.

**Theorem 1.1a** (Stochastic Datko-Pazy Theorem, first version). Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space $E$. The following assertions are equivalent:

(a) For all $x \in E$, $T(\cdot)x \in \gamma(\mathbb{R}_+, E)$.

(b) There exists an $\varepsilon > 0$ such that for all $x \in E$, $t \mapsto e^{\varepsilon t}T(t)x \in \gamma(\mathbb{R}_+, E)$.

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If $E$ is a Hilbert space, $\gamma(\mathbb{R}_+, E) = L^2(\mathbb{R}_+, E)$ and Theorem 1.1a is equivalent to the Datko’s theorem mentioned above.

As explained in [10], $\gamma$–radonifying operators play an important role in the study of the following stochastic abstract Cauchy problem on $E$:

$$(SCP)_{(A,B)} \begin{cases} dU(t) & = & AU(t) \, dt + B \, dW_H(t), & t \geq 0, \\ U(0) & = & 0. & \end{cases}$$

Here, $H$ is a separable Hilbert space, $B \in \mathcal{B}(H, E)$ is a bounded operator, and $W_H$ is an $H$-cylindrical Brownian motion. Theorem 1.1a can be reformulated in terms of invariant measures for $(SCP)_{(A,B)}$ as follows.

**Theorem 1.1b** (Stochastic Datko-Pazy theorem, second version). With the above notations, the following assertions are equivalent:

(a) For all rank one operators $B \in \mathcal{B}(H, E)$, the problem $(SCP)_{(A,B)}$ admits an invariant measure.

(b) There exists an $\varepsilon > 0$ such that for all rank one operators $B \in \mathcal{B}(H, E)$, the problem $(SCP)_{(A+\varepsilon,B)}$ admits an invariant measure.

For unexplained terminology and more information on the stochastic Cauchy problem and invariant measures we refer to [2, 9, 10].

### 2 Riesz bases and $\gamma$-radonifying operators

Let $\mathcal{H}$ be a Hilbert space and $E$ a Banach space. Let $(\gamma_n)_{n \geq 1}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A bounded linear operator $R : \mathcal{H} \rightarrow E$ is called *almost summing* if

$$\|R\|_{\gamma_{\infty}(\mathcal{H}, E)} := \sup_{N \geq 1} \left\| \sum_{n=1}^{N} \gamma_n R h_n \right\|_{L^2(\Omega, E)} < \infty,$$

where the supremum is taken over all $N \in \mathbb{N}$ and all orthonormal systems $\{h_1, \ldots, h_N\}$ in $\mathcal{H}$. Endowed with this norm, the space $\gamma_{\infty}(\mathcal{H}, E)$ of all almost summing operators is a Banach space. Moreover, $\gamma_{\infty}(\mathcal{H}, E)$ is an operator ideal in $\mathcal{B}(\mathcal{H}, E)$. The closure of the finite rank operators in $\gamma_{\infty}(\mathcal{H}, E)$ will be denoted by $\gamma(\mathcal{H}, E)$. Operators belonging to this space are called $\gamma$–radonifying.

Again $\gamma(\mathcal{H}, E)$ is an operator ideal in $\mathcal{B}(\mathcal{H}, E)$.

Let us now assume that $\mathcal{H}$ is a separable Hilbert space. Under this assumption one has $R \in \gamma_{\infty}(\mathcal{H}, E)$ if and only if for some (every) orthonormal basis $(h_n)_{n \geq 1}$ for $\mathcal{H},$

$$M := \sup_{N \geq 1} \left\| \sum_{n=1}^{N} \gamma_n R h_n \right\|_{L^2(\Omega, E)} < \infty.$$

In that case, $\|R\|_{\gamma_{\infty}(\mathcal{H}, E)} = M$. Furthermore, one has $R \in \gamma(\mathcal{H}, E)$ if and only if for some (every) orthonormal basis $(h_n)_{n \geq 1}$ for $\mathcal{H}, \sum_{n \geq 1} \gamma_n R h_n$ converges in $L^2(\Omega, E)$. In that case,

$$\|R\|_{\gamma(\mathcal{H}, E)} = \left\| \sum_{n \geq 1} \gamma_n R h_n \right\|_{L^2(\Omega, E)}.$$
If $E$ does not contain a closed subspace isomorphic to $c_0$, then by a result of Hoffmann-Jørgensen and Kwapien [8, Theorem 9.29], $\gamma(\mathcal{H}, E) = \gamma_{\infty}(\mathcal{H}, E)$.

We will apply the above notions to the space $\mathcal{H} = L^2(\mathbb{R}_+, H)$ where $H$ is a separable Hilbert space. For an operator-valued function $\phi : \mathbb{R}_+ \to \mathcal{B}(H, E)$ which is $H$-strongly measurable in the sense that $t \mapsto \phi(t)h$ is strongly measurable for all $h \in H$, and weakly square integrable in the sense that $t \mapsto \phi^*(t)x^*$ is square Bochner integrable for all $x^* \in E^*$, let $R_\phi \in \mathcal{B}(L^2(\mathbb{R}_+, H), E)$ be defined as the Pettis integral operator

$$R_\phi(f) := \int_{\mathbb{R}_+} \phi(t)f(t) \, dt.$$ 

We say that $\phi \in \gamma(\mathbb{R}_+, H, E)$ if $R_\phi \in \gamma(L^2(\mathbb{R}_+, H), E)$ and write

$$\|\phi\|_{\gamma(\mathbb{R}_+, H, E)} := \|R_\phi\|_{\gamma(L^2(\mathbb{R}_+, H), E)}.$$

If $H = \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is the underlying scalar field, we write $\gamma(\mathbb{R}_+, E)$ for $\gamma(\mathbb{R}_+, H, E)$. For almost summing operators we use an analogous notation.

For more information we refer to [4, 6, 9, 10].

**Hilbert and Bessel sequences.** Let $\mathcal{H}$ be a Hilbert space and $I \subseteq \mathbb{Z}$ an index set. A sequence $(h_i)_{i \in I}$ in $\mathcal{H}$ is said to be a *Hilbert sequence* if there exists a constant $C > 0$ such that for all scalars $(\alpha_i)_{i \in I}$,

$$\left( \left\| \sum_{i \in I} \alpha_i h_i \right\|^2 \right)^{\gamma_2} \leq C \left( \sum_{i \in I} |\alpha_i|^2 \right)^{\gamma_2}.$$ 

The infimum of all admissible constants $C > 0$ will be denoted by $C_H(\{h_i : i \in I\})$. A Hilbert sequence that is a Schauder basis is called a *Hilbert basis* (cf. [14, Section 1.8]).

The sequence $(h_i)_{i \in I}$ is said to be a *Bessel sequence* if there exists a constant $c > 0$ such that for all scalars $(\alpha_i)_{i \in I}$,

$$c \left( \sum_{i \in I} |\alpha_i|^2 \right)^{\gamma_2} \leq \left( \left\| \sum_{i \in I} \alpha_i h_i \right\|^2 \right)^{\gamma_2}.$$ 

The supremum of all admissible constants $c > 0$ will be denoted by $C_B(\{h_i : i \in I\})$. Notice that every Bessel sequence is linearly independent. A Bessel sequence that is a Schauder basis is called a *Bessel basis*. A sequence $(h_i)_{i \in I}$ that is a Bessel sequence and a Hilbert sequence is said to be a *Riesz sequence*. A sequence $(h_i)_{i \in I}$ that is a Bessel basis and a Hilbert basis is said to be a *Riesz basis* (cf. [14, Section 1.8]).

In the above situation if it is clear which sequence in $\mathcal{H}$ we refer to, we use the short-hand notation $C_H$ and $C_B$ for $C_H(\{h_i : i \in I\})$ and $C_B(\{h_i : i \in I\})$.

In the next results we study the relation between $\gamma$-radonifying operators and Hilbert and Bessel sequences.

**Proposition 2.1.** Let $(f_n)_{n \geq 1}$ be a Hilbert sequence in $\mathcal{H}$.

(a) If $R \in \gamma_{\infty}(\mathcal{H}, E)$, then

$$\sup_{N \geq 1} \left\| \sum_{n=1}^N \gamma_n Rf_n \right\|_{L^2(\Omega, E)} \leq C_H \|R\|_{\gamma_{\infty}(\mathcal{H}, E)}. \quad (1)$$ 


(b) If \( R \in \gamma(\mathcal{H}, E) \), then \( \sum_{n \geq 1} \gamma_n R f_n \) converges in \( L^2(\Omega, E) \) and

\[
\left\| \sum_{n \geq 1} \gamma_n R f_n \right\|_{L^2(\Omega, E)} \leq C_H \| R \|_{\gamma(\mathcal{H}, E)}.
\]

Proof. (a): Fix \( N \geq 1 \) and let \( \{h_1, \ldots, h_N\} \) be an orthonormal system in \( \mathcal{H} \). Since \( (f_n)_{n \geq 1} \) is a Hilbert sequence there is a unique \( T \in \mathcal{B}(\mathcal{H}) \) such that \( Th_n = f_n \) for \( n = 1, \ldots, N \) and \( Tx = 0 \) for all \( x \in \{h_1, \ldots, h_N\}^\perp \). Moreover, \( \| T \| \leq C_H \). By the right ideal property we have \( R \circ T \in \gamma_\infty(\mathcal{H}, E) \) and, for all \( N \geq 1 \),

\[
\left\| \sum_{n=1}^N \gamma_n R f_n \right\|_{L^2(\Omega, E)} = \left\| \sum_{n=1}^N \gamma_n R T h_n \right\|_{L^2(\Omega, E)} \leq \| R \circ T \|_{\gamma_\infty(\mathcal{H}, E)} \leq C_H \| R \|_{\gamma_\infty(\mathcal{H}, E)}.
\]

(b): This is proved in a similar way. \( \square \)

**Proposition 2.2.** Let \( (f_n)_{n \geq 1} \) be a Bessel sequence in \( \mathcal{H} \) and let \( \mathcal{H}_f \) denote its closed linear span.

(a) If \( \sup_{N \geq 1} \left\| \sum_{n=1}^N \gamma_n R f_n \right\|_{L^2(\Omega, E)} < \infty \), then \( R \in \gamma_\infty(\mathcal{H}_f, E) \) and

\[
\| R \|_{\gamma_\infty(\mathcal{H}_f, E)} \leq C_B^{-1} \sup_{N \geq 1} \left\| \sum_{n=1}^N \gamma_n R f_n \right\|_{L^2(\Omega, E)}.
\]

(b) If \( \sum_{n \geq 1} \gamma_n R f_n \) converges in \( L^2(\Omega, E) \), then \( R \in \gamma(\mathcal{H}_f, E) \) and

\[
\| R \|_{\gamma(\mathcal{H}_f, E)} \leq C_B^{-1} \left\| \sum_{n \geq 1} \gamma_n R f_n \right\|_{L^2(\Omega, E)}.
\]

Proof. Let \( (h_n)_{n \geq 1} \) an orthonormal basis for \( \mathcal{H}_f \). Since \( (f_n)_{n \geq 1} \) is a Bessel sequence there is a unique \( T \in \mathcal{B}(\mathcal{H}, E) \) such that \( Tf_n = h_n \) and \( Tx = 0 \) for \( x \in \mathcal{H}_f^\perp \). Notice that \( \| T \| \leq C_B^{-1} \). On the linear span \( \mathcal{H}_0 \) of the sequence \( (f_n)_{n \geq 1} \) we define an inner product by \( \langle x, y \rangle_T := \langle Tx, Ty \rangle_{\mathcal{H}} \). Note that this is well defined by the linear independence of the sequence \( (f_n)_{n \geq 1} \). Let \( \mathcal{H}_T \) denote the Hilbert space completion of \( \mathcal{H}_0 \) with respect to \( \langle \cdot, \cdot \rangle_T \). The identity mapping on \( \mathcal{H}_f \) extends to a bounded operator \( j : \mathcal{H}_f \to \mathcal{H}_T \) with norm \( \| j \| \leq C_B^{-1} \). Clearly, \( (j f_n)_{n \geq 1} \) is an orthonormal sequence in \( \mathcal{H}_T \) with dense span, and therefore it is an orthonormal basis for \( \mathcal{H}_T \). It is elementary to verify that the assumption on \( R \) may now be translated as saying that \( R \) extends in a unique way to an almost summing operator (in part (a)), respectively a \( \gamma \)-radonifying operator (in part (b)), denoted by \( R_T \), from \( \mathcal{H}_T \) to \( E \). We estimate

\[
\left\| \sum_{n \geq 1} \alpha_n j h_n \right\|_{\mathcal{H}_T} = \left\| \sum_{n \geq 1} \alpha_n Th_n \right\|_{\mathcal{H}} \leq C_B^{-1} \left\| \sum_{n \geq 1} \alpha_n h_n \right\|_{\mathcal{H}} = C_B^{-1} \left( \sum_{n \geq 1} |\alpha_n|^2 \right)^{1/2}.
\]

From this we deduce that \( (j h_n)_{n \geq 1} \) is a Hilbert sequence in \( \mathcal{H}_T \) with constant \( \leq C_B^{-1} \). Hence we may apply Proposition 2.1 to the operator \( R_T : \mathcal{H}_T \to E \) and the Hilbert sequence \( (j h_n)_{n \geq 1} \) in \( \mathcal{H}_T \) to obtain the result. \( \square \)
As a consequence of the above results we obtain:

**Theorem 2.3.** Let \((f_n)_{n \geq 1}\) be a Riesz basis in the Hilbert space \(\mathcal{H}\).

(a) One has \(R \in \gamma_\infty(\mathcal{H}, E)\) if and only if \(\sup_{N \geq 1} \left\| \sum_{n=1}^{N} \gamma_n R f_n \right\|_{L^2(\Omega, E)} < \infty\). In that case (1) and (3) hold.

(b) One has \(R \in \gamma(\mathcal{H}, E)\) if and only if \(\sum_{n \geq 1} \gamma_n R f_n\) converges in \(L^2(\Omega, E)\). In that case (2) and (4) hold.

The following well-known lemma identifies a class of Riesz sequences in \(L^2(\mathbb{R})\). For convenience we include the short proof from [1, Theorem 2.1]. Let \(T\) be the unit circle in \(\mathbb{C}\).

**Lemma 2.4.** Let \(f \in L^2(\mathbb{R})\) and define the sequence \((f_n)_{n \in \mathbb{Z}}\) in \(L^2(\mathbb{R})\) by \(f_n(t) = e^{2\pi i n t} f(t)\). Define \(F : T \to \mathbb{R}\) as

\[
F(e^{2\pi i t}) := \sum_{k \in \mathbb{Z}} |f(t + k)|^2
\]

(a) The sequence \((f_n)_{n \in \mathbb{Z}}\) is a Bessel sequence in \(L^2(\mathbb{R})\) if and only if there exists a constant \(A > 0\) such that \(A \leq F(e^{2\pi i t})\) for almost all \(t \in [0, 1]\).

(b) The sequence \((f_n)_{n \in \mathbb{Z}}\) is a Hilbert sequence in \(L^2(\mathbb{R})\) if and only if there exists a constant \(B > 0\) such that \(F(e^{2\pi i t}) \leq B\) for almost all \(t \in [0, 1]\).

In these cases, \(C_B^2 = \text{ess inf} F\) and \(C_H^2 = \text{ess sup} F\) respectively.

**Proof.** Both assertions are obtained by observing that for \(I \subseteq \mathbb{Z}\) and \((a_n)_{n \in I}\) in \(\mathbb{C}\) we may write

\[
\left\| \sum_{n \in I} a_n f_n \right\|_{L^2(\mathbb{R})}^2 = \sum_{k \in \mathbb{Z}} \int_{I} \left| \sum_{n \in I} a_n e^{2\pi i n t} f(t) \right|^2 dt = \int_{0}^{1} \left| \sum_{n \in I} a_n e^{2\pi i n t} \right|^2 F(e^{2\pi i t}) dt.
\]

The following application of Lemma 2.4 will be used below.

**Example 2.5.** Let \(\rho \in [0, 1)\) and \(a > 0\). For \(n \in \mathbb{Z}\) let

\[
f_n(t) = e^{-at + 2\pi(n + \rho)it} \mathbb{1}_{[0, \infty)}(t).
\]

Then \((f_n)_{n \in \mathbb{Z}}\) is a Riesz sequence in \(L^2(\mathbb{R})\) with constants \(C_B^2 = \frac{e^{-2\alpha}}{e^{-2\alpha} - 1}\) and \(C_H^2 = \frac{e^{-2\alpha}}{e^{-2\alpha} - 1}\). Indeed, let \(f(t) := e^{-at + 2\pi \rho it} \mathbb{1}_{[0, \infty)}(t)\). For all \(t \in [0, 1]\),

\[
F(e^{2\pi i t}) = \sum_{k \in \mathbb{Z}} |f(t + k)|^2 = \sum_{k=0}^{\infty} e^{-2a(t + k)} = \frac{e^{2a(1-t)}}{e^{2a} - 1}.
\]

Now Lemma 2.4 implies the result.
3 Main results

In this section we use Proposition 2.1 to obtain an alternative proof of [10, Theorem 3.4] on the $R$-boundedness of certain Laplace transforms. This result is applied to strongly continuous semigroups to obtain estimates for the absissa of $R$-boundedness of the resolvent. From this we deduce Theorem 1.1a as well as bounded perturbation results for the existence of solutions and invariant measures for the problem $(SCP)_{(A,B)}$.

Let $(r_n)_{n \geq 1}$ be a Rademacher sequence on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A family of operators $\mathcal{F} \subseteq \mathcal{B}(E)$ is called $R$-bounded if there exists a constant $C > 0$ such that for all $N \geq 1$ and all sequences $(T_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ and $(x_n)_{n=1}^N \subseteq E$ we have

$$E \left\| \sum_{n=1}^N r_n T_n x_n \right\|^2 \leq C^2 E \left\| \sum_{n=1}^N r_n x_n \right\|^2.$$  

The least possible constant $C$ is called the $R$-bound of $\mathcal{F}$, notation $\mathcal{R}(\mathcal{F})$. Clearly, every $R$-bounded family $\mathcal{F}$ is uniformly bounded and sup$_{T \in \mathcal{F}} \|T\| \leq \mathcal{R}(\mathcal{F})$.

Following [10], for an operator $T \in \mathcal{B}(L^2(\mathbb{R}_+), E)$ we define the Laplace transform $\hat{T} : \{\lambda \in \mathbb{C} : \text{Re} \lambda > 0\} \to E$ as

$$\hat{T}(\lambda) := Te^\lambda.$$ 

Here $e_\lambda \in L^2(\mathbb{R}_+)$ is given by $e_\lambda(t) = e^{-\lambda t}$. For a Banach space $F$ and a bounded operator $\Theta : F \to \mathcal{B}(L^2(\mathbb{R}_+), E)$ we define the Laplace transform $\hat{\Theta} : \{\lambda \in \mathbb{C} : \text{Re} \lambda > 0\} \to \mathcal{B}(F, E)$ as

$$\hat{\Theta}(\lambda)y := \hat{\Theta}y(\lambda) \quad \text{Re} \lambda > 0, \ y \in F.$$ 

The following result is a slight refinement of [10, Theorem 3.4]. The main novelty is the simple proof of the estimate (5).

**Theorem 3.1.** Let $F$ be a Banach space. Let $\Theta : F \to \gamma_\infty(L^2(\mathbb{R}_+), E)$ be a bounded operator and let $\delta > 0$. Then $\hat{\Theta}$ is $R$-bounded on the half-plane $\{\lambda \in \mathbb{C} : \text{Re} \lambda > \delta\}$ and there exists a universal constant $C$ such that

$$\mathcal{R} \left( \{\hat{\Theta}(\lambda) : \text{Re} \lambda \geq \delta\} \right) \leq \|\Theta\| \frac{C}{\sqrt{\delta}}.$$ 

*Proof.* Let $\delta > 0$. Consider the set $\{\lambda \in \mathbb{C} : \text{Re} \lambda = \delta\}$. Fix $\sigma \in [\delta, \frac{3}{2}\delta]$ and $\rho \in [0, 1)$. For $n \in \mathbb{Z}$ let $g_n : \mathbb{R}_+ \to \mathbb{C}$ be given by

$$g_n(t) = e^{-\sigma t + (n+\rho)\delta t}.$$ 

By a substitution, this reduces to Example 2.5, whence $(g_n)_{n \geq 1}$ is a Riesz sequence in $L^2(\mathbb{R}_+)$ with constant $0 < C_H \leq (\frac{C}{\delta})^{1/2}$ where $C := 2\pi \frac{e^{\pi^2/4}}{e^{\pi/2}-1}$. For $y \in F$, we may apply Proposition 2.1 to obtain

$$\left\| \sum_{n=-N}^N \gamma_n \hat{\Theta}(\sigma - (n+\rho)\delta i)y \right\|_{L^2(\Omega, E)} \leq \left\| \sum_{n=-N}^N \gamma_n (\Theta y)g_n \right\|_{L^2(\Omega, E)} \leq C_H \|\Theta y\| \|y\|_{\gamma_\infty(\Omega, E)} \leq \left(\frac{C}{\delta}\right)^{1/2} \|\Theta\| \|y\|.$$  

The rest of the proof follows the lines in [10].
In what follows we let \((T(t))_{t \geq 0}\) be a strongly continuous semigroup on \(E\) with generator \(A\). We recall from \cite{9, 10} that the problem (SCP) \((A, B)\) admits a (unique) solution if and only if \(T(\cdot)B\) belongs to \(\gamma([0, T], H, E)\) for some (all) \(T > 0\). Furthermore, an invariant measure exists if and only if \(T(\cdot)B\) belongs to \(\gamma(\mathbb{R}_+, H, E)\).

The next theorem improves \cite[Theorem 1.3]{10}, where the bound \(s_R(A) \leq 0\) was obtained.

**Theorem 3.2.** Assume that for all \(x \in E\), \(T(\cdot)x \in \gamma_{\infty}(\mathbb{R}_+, E)\). Then \(s_R(A) < 0\), i.e., there exists an \(\varepsilon > 0\) such that \(\{R(\lambda, A) : \Re \lambda \geq -\varepsilon\}\) is \(R\)-bounded.

**Proof.** By the closed graph theorem there exists an \(M > 0\) such that \(\|T(\cdot)x\|_{\gamma_{\infty}(\mathbb{R}_+, E)} \leq M\|x\|\). By Theorem 3.1, \(\{\lambda \in \mathbb{C} : \Re \lambda > 0\} \subseteq \varrho(A)\) and

\[
\mathcal{R}(\{R(\lambda, A) : \Re \lambda \geq \delta\}) \leq \frac{c}{\sqrt{\delta}}
\]  

(6)

for all \(\delta > 0\), where \(c := CM\) with \(C\) the universal constant of Theorem 3.1. The following standard argument shows that this implies the bound

\[
s(A) \leq -\frac{1}{4c^2}.
\]  

(7)

Choose \(\delta > 0\) and let \(\mu \in \sigma(A)\) be such that \(\Re \mu > s(A) - \delta\). With \(\lambda = \frac{1}{4c^2} + i \Im \mu\) it follows that

\[
\frac{1}{4c^2} - s(A) + \delta \geq \text{dist}(\lambda, \sigma(A)) \geq \frac{1}{\|R(\lambda, A)\|} \geq \frac{\sqrt{\Re \lambda}}{c} = \frac{1}{2c^2}.
\]

Thus \(s(A) \leq -\frac{1}{4c^2} + \delta\). Since \(\delta > 0\) was arbitrary, this gives (7).

Now let \(\varepsilon_0 := \frac{1}{4c^2}\). For \(\lambda\) with \(-\varepsilon_0 < \Re \lambda < 3\varepsilon_0\) we may write

\[
R(\lambda, A) = \sum_{n \geq 0} (\varepsilon_0 - \Re \lambda)^n R(\varepsilon_0 + i \Im \lambda, A)^{n+1}.
\]

Fix \(0 < \varepsilon < \varepsilon_0\). We claim that \(\{R(\lambda, A) : \Re \lambda = -\varepsilon\}\) is \(R\)-bounded. To see this let \((r_k)_{k=1}^K\) be a Rademacher sequence on \((\Omega, \mathcal{F}, \mathbb{P})\), let \((\lambda_k)_{k=1}^K\) be such that \(\Re \lambda_k = -\varepsilon\), and let \((x_k)_{k=1}^K\) be a sequence in \(E\). We may estimate

\[
\left\| \sum_{k=1}^K r_k R(\lambda_k, A)x_k \right\|_{L^2(\Omega, E)} = \left\| \sum_{n \geq 0} \sum_{k=1}^K r_k (\varepsilon_0 + \varepsilon)^n R(\varepsilon_0 + i \Im \lambda_k, A)^{n+1} x_k \right\|_{L^2(\Omega, E)}
\]

\[
\leq \sum_{n \geq 0} (\varepsilon_0 + \varepsilon)^n \left\| \sum_{k=1}^K r_k R(\varepsilon_0 + i \Im \lambda_k, A)^{n+1} x_k \right\|_{L^2(\Omega, E)}
\]

\[
\leq \sum_{n \geq 0} (\varepsilon_0 + \varepsilon)^n \left( \frac{c}{\varepsilon_0} \right)^{n+1} \left\| \sum_{k=1}^K r_k x_k \right\|_{L^2(\Omega, E)}
\]

\[
= \frac{1}{\varepsilon_0 - \varepsilon} \left\| \sum_{k=1}^K r_k x_k \right\|_{L^2(\Omega, E)},
\]

where we used that \(\varepsilon_0 = \frac{1}{4c^2}\). This proves the claim. Now the result is obtained via \cite[Proposition 2.8]{13}. \(\square\)
As an application of Theorem 3.2 we have the following bounded perturbation result for the existence of a solution for the perturbed problem.

**Theorem 3.3.** Let $P \in \mathcal{B}(E)$ and $B \in \mathcal{B}(H, E)$. If $(SCP)_{(A,B)}$ has a solution, then $(SCP)_{(A+P,B)}$ has a solution as well.

*Proof.* For $\omega \in \mathbb{R}$ denote $A_\omega = A - \omega$ and $T_\omega(\cdot) := e^{-\omega T(\cdot)}$. It follows from [10, Proposition 4.5] that for all $\omega > \omega_0(A)$, $T_\omega(\cdot) \in \gamma(\mathbb{R}_+, H, E)$. From [7, Corollary 2.17] it follows that for all $\omega > \omega_0(A) + 1$,

$$\mathcal{R} \left( \{ R(\lambda, A_\omega) : \text{Re} \lambda \geq 0 \} \right) \leq \frac{c}{\omega - \omega_0(A) - 1},$$

where $c$ is a constant depending only on $T$. Choose $\omega_1 > \omega_0(A) + 1$ so large that $\frac{c}{\omega_1 - \omega_0(A) - 1} < 1$. By [10, Lemma 5.1], $R(i, A_\omega) \in \gamma(\mathbb{R}_+, H, E)$.

Denote by $(S(t))_{t \geq 0}$ the semigroup generated by $A+P$ (cf. [5, Section III.1] or [12, Chapter III]) and let $S_{\omega_1}(t) := e^{-\omega_1 t}S(t), t \geq 0$. Since

$$\mathcal{R} \left( \{ R(is, A_\omega) : s \in \mathbb{R} \} \right) \leq \mathcal{R} \left( \{ R(is, A_\omega) : s \in \mathbb{R} \} \right),$$

it follows from $iR \subseteq \mathfrak{g}(A_\omega)$ that $iR \subseteq \mathfrak{g}(A_\omega + P)$ and

$$R(is, A_\omega + P)B = \sum_{n=0}^{\infty} (R(is, A_\omega)) P^n R(is, A_\omega)B =: R_{A_\omega, P, \omega_1}(s)R(is, A_\omega)B.$$

Moreover, as in Theorem 3.2, and using the fact that $C < 1$, $\{ R_{A_\omega, P, \omega_1}(s) : s \in \mathbb{R} \}$ is $R$-bounded with constant $\frac{1}{1-C}$. From [6, Proposition 4.11] we deduce that

$$\| R(i, A_\omega + P)B \|_{\gamma(\mathbb{R}, H, E)} \leq \frac{1}{1-C} \| R(i, A_\omega)B \|_{\gamma(\mathbb{R}, H, E)}.$$ 

Now [10, Lemma 5.1] shows that $S_{\omega_1}(\cdot)B \in \gamma(\mathbb{R}_+, H, E)$. It follows from the right ideal property that for all $t > 0$,

$$\| S(\cdot)B \|_{\gamma(0, t, H, E)} \leq e^{t \omega_1} \| S_{\omega_1}(\cdot)B \|_{\gamma(0, 1, H, E)}$$

and the result can be obtained via [9, Theorem 7.1]. \hfill \Box

Concerning existence and uniqueness of invariant measures we obtain:

**Theorem 3.4.** Assume that $s(A) < 0$ and that $\{ R(is, A) : s \in \mathbb{R} \}$ is $R$-bounded. Let $B \in \mathcal{B}(H, E)$ such that $(SCP)_{(A,B)}$ admits an invariant measure. Then there exists a $\delta > 0$ such that for all $P \in \mathcal{B}(E)$ with $\| P \| < \delta$, $(SCP)_{(A+P,B)}$ admits a unique invariant measure.

*Proof.* Let $\delta > 0$ such that $\mathcal{R} \left( \{ R(is, A) : s \in \mathbb{R} \} \right) \leq \frac{1}{\delta}$. Then, if $\| P \| < \delta$

$$\mathcal{R} \left( \{ R(is, A) : s \in \mathbb{R} \} \right) \leq \mathcal{R} \left( \{ R(is, A) : s \in \mathbb{R} \} \right).$$

As in Theorem 3.3 it can be deduced that

$$\| R(i, A+P)B \|_{\gamma(\mathbb{R}, H, E)} \leq \frac{1}{1-C} \| R(i, A)B \|_{\gamma(\mathbb{R}, H, E)}.$$ 

The existence of an invariant measure now follows from [10, Proposition 4.4 and Lemma 5.1].

By [10, Corollary 4.3], for uniqueness it suffices to note that $R(\lambda, A+P)$ is uniformly bounded for Re$\lambda > 0$. \hfill \Box
In particular, the $R$-boundedness of $\{R(is, A) : s \in \mathbb{R}\}$ implies that an invariant measure for $(SCP)_{(A, B)}$, if one exists, is unique. On the other hand, if $i\mathbb{R} \subseteq \sigma(A)$ but $\{R(is, A) : s \in \mathbb{R}\}$ fails to be $R$-bounded, then Theorem 3.2 shows that there exists a rank one operator $B' \in \mathcal{B}(H, E)$ such that the problem $(SCP)_{(A, B')}$ fails to have an invariant measure. As a result we obtain that if $(SCP)_{(A, B')}$ fails to have a unique invariant measure, then there exists a rank one operator $B' \in \mathcal{B}(H, E)$ such that the problem $(SCP)_{(A, B')}$ fails to have an invariant measure.

Proof of Theorems 1.1a and 1.1b. If $T(\cdot)x \in \gamma(\mathbb{R}_+, E)$ for all $x \in E$, then by Theorem 3.2 $s(A) < 0$ and $\{R(is, A) : s \in \mathbb{R}\}$ is $R$-bounded. Thus, Theorem 3.4 applies to the bounded perturbation $P = \delta \cdot I_E$. □

References


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