A stochastic Datko-Pazy theorem

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Let $H$ be a Hilbert space and $E$ a Banach space. In this note we present a sufficient condition for an operator $R: H \to E$ to be $\gamma$–radonifying in terms of Riesz sequences in $H$. This result is applied to recover a result of Lutz Weis and the second named author on the $R$-boundedness of resolvents, which is used to obtain a Datko-Pazy type theorem for the stochastic Cauchy problem. We also present some perturbation results.

1 Introduction

The well-known Datko-Pazy theorem states that if $(T(t))_{t \geq 0}$ is a strongly continuous semigroup on a Banach space $E$ such that all orbits $T(\cdot)x$ belong to the space $L^p(\mathbb{R}_+, E)$ for some $p \in [1, \infty)$, then $(T(t))_{t \geq 0}$ is uniformly exponentially stable, or equivalently, there exists an $\varepsilon > 0$ such that all orbits $t \mapsto e^{\varepsilon t}T(t)x$ belong to $L^p(\mathbb{R}_+, E)$. For $p = 2$ and Hilbert spaces $E$ this result is due to Datko [3], and the general case was obtained by Pazy [14].

In this note we prove a stochastic version of the Datko-Pazy theorem for spaces of $\gamma$–radonifying operators (cf. Section 2). Let us denote by $\gamma(\mathbb{R}_+, E)$ the space of all strongly measurable functions $\phi: \mathbb{R}_+ \to E$ for which the integral operator

$$f \mapsto \int_0^\infty f(t)\phi(t) \, dt$$

is well-defined and $\gamma$-radonifying from $L^2(\mathbb{R}_+)$ to $E$.

**Theorem 1.1a** (Stochastic Datko-Pazy Theorem, first version). *Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space $E$. The following assertions are equivalent:

(a) For all $x \in E$, $T(\cdot)x \in \gamma(\mathbb{R}_+, E)$.

(b) There exists an $\varepsilon > 0$ such that for all $x \in E$, $t \mapsto e^{\varepsilon t}T(t)x \in \gamma(\mathbb{R}_+, E)$.

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If $E$ is a Hilbert space, $\gamma(\mathbb{R}_+, E) = L^2(\mathbb{R}_+, E)$ and Theorem 1.1a is equivalent to the Datko’s theorem mentioned above.

As explained in [12], $\gamma$-radonifying operators play an important role in the study of the following stochastic abstract Cauchy problem on $E$:

\[
\begin{aligned}
(\text{SCP})_{(A,B)} & \quad \begin{cases}
  dU(t) = AU(t) \, dt + B \, dW_H(t), & t \geq 0, \\
  U(0) = 0.
\end{cases}
\end{aligned}
\]

Here, $H$ is a separable Hilbert space, $B \in \mathcal{B}(H, E)$ is a bounded operator, and $W_H$ is an $H$-cylindrical Brownian motion. Theorem 1.1a can be reformulated in terms of invariant measures for $(\text{SCP})_{(A,B)}$ as follows.

**Theorem 1.1b** (Stochastic Datko-Pazy theorem, second version). With the above notations, the following assertions are equivalent:

(a) For all rank one operators $B \in \mathcal{B}(H, E)$, the problem $(\text{SCP})_{(A,B)}$ admits an invariant measure.

(b) There exists an $\varepsilon > 0$ such that for all rank one operators $B \in \mathcal{B}(H, E)$, the problem $(\text{SCP})_{(A+\varepsilon,B)}$ admits an invariant measure.

For unexplained terminology and more information on the stochastic Cauchy problem and invariant measures we refer to [2, 11, 12].

### 2 Riesz bases and $\gamma$-radonifying operators

Let $\mathcal{H}$ be a Hilbert space and $E$ a Banach space. Let $(\gamma_n)_{n \geq 1}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A bounded linear operator $R : \mathcal{H} \to E$ is called *almost summing* if

\[
\|R\|_{\gamma(\mathcal{H}, E)} := \sup \left\| \sum_{n=1}^{N} \gamma_n R h_n \right\|_{L^2(\Omega, E)} < \infty,
\]

where the supremum is taken over all $N \in \mathbb{N}$ and all orthonormal systems $\{h_1, \ldots, h_N\}$ in $\mathcal{H}$. Endowed with this norm, the space $\gamma(\mathcal{H}, E)$ of all almost summing operators is a Banach space. Moreover, $\gamma(\mathcal{H}, E)$ is an operator ideal in $\mathcal{B}(\mathcal{H}, E)$. The closure of the finite rank operators in $\gamma(\mathcal{H}, E)$ will be denoted by $\gamma(\mathcal{H}, E)$. Operators belonging to this space are called *$\gamma$-radonifying*. Again $\gamma(\mathcal{H}, E)$ is an operator ideal in $\mathcal{B}(\mathcal{H}, E)$.

Let us now assume that $\mathcal{H}$ is a separable Hilbert space. Under this assumption one has $R \in \gamma(\mathcal{H}, E)$ if and only if for some (every) orthonormal basis $(h_n)_{n \geq 1}$ for $\mathcal{H}$,

\[
M := \sup_{N \geq 1} \left\| \sum_{n=1}^{N} \gamma_n R h_n \right\|_{L^2(\Omega, E)} < \infty.
\]

In that case, $\|R\|_{\gamma(\mathcal{H}, E)} = M$. Furthermore, one has $R \in \gamma(\mathcal{H}, E)$ if and only if for some (every) orthonormal basis $(h_n)_{n \geq 1}$ for $\mathcal{H}$, $\sum_{n \geq 1} \gamma_n R h_n$ converges in $L^2(\Omega, E)$. In that case,

\[
\|R\|_{\gamma(\mathcal{H}, E)} = \left\| \sum_{n \geq 1} \gamma_n R h_n \right\|_{L^2(\Omega, E)}.
\]
If $E$ does not contain a closed subspace isomorphic to $c_0$, then by a result of Hoffmann-Jorgensen and Kwapien [10, Theorem 9.29], $\gamma(\mathcal{H}, E) = \gamma_\infty(\mathcal{H}, E)$.

We will apply the above notions to the space $\mathcal{H} = L^2(\mathbb{R}_+, H)$ where $H$ is a separable Hilbert space. For an operator-valued function $\phi : \mathbb{R}_+ \to \mathcal{B}(H, E)$ we will apply the above notions to the space $\mathcal{B}(\mathbb{R}_+, H, E)$ for an operator-valued function $H\mapsto R_\phi$ for an operator-valued function $H\mapsto R_\phi$.

The infimum of all admissible constants $C > 0$ will be denoted by $C_H(\{h_i : i \in I\})$. A Hilbert sequence that is a Schauder basis is called a Hilbert basis (cf. [17, Section 1.8]).

The sequence $(h_i)_{i \in I}$ is said to be a Bessel sequence if there exists a constant $c > 0$ such that for all scalars $(\alpha_i)_{i \in I}$,

$$
\left( \left\| \sum_{i \in I} \alpha_i h_i \right\|^2 \right)^{\gamma_2} \leq C \left( \sum_{i \in I} |\alpha_i|^2 \right)^{\gamma_2}.
$$

The supremum of all admissible constants $c > 0$ will be denoted by $C_B(\{h_i : i \in I\})$. Notice that every Bessel sequence is linearly independent. A Bessel sequence that is a Schauder basis is called a Bessel basis. A sequence $(h_i)_{i \in I}$ that is a Bessel sequence and a Hilbert sequence is said to be a Riesz sequence. A sequence $(h_i)_{i \in I}$ that is a Bessel basis and a Hilbert basis is said to be a Riesz basis (cf. [17, Section 1.8]).

In the above situation if it is clear which sequence in $\mathcal{H}$ we refer to, we use the short-hand notation $C_H$ and $C_B$ for $C_H(\{h_i : i \in I\})$ and $C_B(\{h_i : i \in I\})$.

In the next results we study the relation between $\gamma$-radonifying operators and Hilbert and Bessel sequences.

**Proposition 2.1.** Let $(f_n)_{n \geq 1}$ be a Hilbert sequence in $\mathcal{H}$.

(a) If $R \in \gamma_\infty(\mathcal{H}, E)$, then

$$
\sup_{N \geq 1} \left\| \sum_{n=1}^N \gamma_n R f_n \right\|_{L^2(\Omega, E)} \leq C_H \left\| R \right\|_{\gamma_\infty(\mathcal{H}, E)}.
$$

(b) If $R \in \gamma_\infty(\mathcal{H}, E)$, then

$$
\sup_{N \geq 1} \left( \sum_{n=1}^N \gamma_n R f_n \right) \leq C_B \left\| R \right\|_{\gamma_\infty(\mathcal{H}, E)}.
$$
(b) If \( R \in \gamma(\mathcal{H}, E) \), then \( \sum_{n \geq 1} \gamma_n Rf_n \) converges in \( L^2(\Omega, E) \) and
\[
\left\| \sum_{n \geq 1} \gamma_n Rf_n \right\|_{L^2(\Omega, E)} \leq C_H \| R \|_{\gamma(\mathcal{H}, E)}.
\] (2)

Proof. (a): Fix \( N \geq 1 \) and let \( \{h_1, \ldots, h_N\} \) be an orthonormal system in \( \mathcal{H} \). Since \( (f_n)_{n \geq 1} \) is a Hilbert sequence there is a unique \( T \in \mathcal{B}(\mathcal{H}) \) such that \( Th_n = f_n \) for \( n = 1, \ldots, N \) and \( Tx = 0 \) for all \( x \in \{h_1, \ldots, h_N\}^\perp \). Moreover, \( \|T\| \leq C_H \). By the right ideal property we have \( R \circ T \in \gamma_\infty(\mathcal{H}, E) \) and, for all \( N \geq 1 \),
\[
\left\| \sum_{n=1}^{N} \gamma_n Rf_n \right\|_{L^2(\Omega, E)} = \left\| \sum_{n=1}^{N} \gamma_n RT h_n \right\|_{L^2(\Omega, E)} \leq \|R \circ T\|_{\gamma_\infty(\mathcal{H}, E)} \leq C_H \|R\|_{\gamma_\infty(\mathcal{H}, E)}.
\]
(b): This is proved in a similar way.

Proposition 2.2. Let \( (f_n)_{n \geq 1} \) be a Bessel sequence in \( \mathcal{H} \) and let \( \mathcal{H}_f \) denote its closed linear span.

(a) If \( \sup_{N \geq 1} \left\| \sum_{n=1}^{N} \gamma_n Rf_n \right\|_{L^2(\Omega, E)} < \infty \), then \( R \in \gamma_\infty(\mathcal{H}_f, E) \) and
\[
\|R\|_{\gamma_\infty(\mathcal{H}_f, E)} \leq C_B^{-1} \sup_{N \geq 1} \left\| \sum_{n=1}^{N} \gamma_n Rf_n \right\|_{L^2(\Omega, E)}.
\] (3)

(b) If \( \sum_{n \geq 1} \gamma_n Rf_n \) converges in \( L^2(\Omega, E) \), then \( R \in \gamma(\mathcal{H}_f, E) \) and
\[
\|R\|_{\gamma(\mathcal{H}_f, E)} \leq C_B^{-1} \left\| \sum_{n \geq 1} \gamma_n Rf_n \right\|_{L^2(\Omega, E)}.
\] (4)

Proof. Let \( (h_n)_{n \geq 1} \) an orthonormal basis for \( \mathcal{H}_f \). Since \( (f_n)_{n \geq 1} \) is a Bessel sequence there is a unique \( T \in \mathcal{B}(\mathcal{H}, E) \) such that \( Tf_n = h_n \) and \( Tx = 0 \) for \( x \in \mathcal{H}_f^\perp \). Notice that \( \|T\| \leq C_B^{-1} \). On the linear span \( \mathcal{H}_0 \) of the sequence \( (f_n)_{n \geq 1} \) we define an inner product by \( [x, y]_T := [Tx, Ty]_\mathcal{H} \). Note that this is well defined by the linear independence of the sequence \( (f_n)_{n \geq 1} \). Let \( \mathcal{H}_T \) denote the Hilbert space completion of \( \mathcal{H}_0 \) with respect to \( [\cdot, \cdot]_T \). The identity mapping on \( \mathcal{H}_f \) extends to a bounded operator \( j : \mathcal{H}_f \hookrightarrow \mathcal{H}_T \) with norm \( \|j\| \leq C_B^{-1} \). Clearly, \( (jh_n)_{n \geq 1} \) is an orthonormal sequence in \( \mathcal{H}_T \) with dense span, and therefore it is an orthonormal basis for \( \mathcal{H}_T \). It is elementary to verify that the assumption on \( R \) may now be translated as saying that \( R \) extends in a unique way to an almost summing operator (in part (a)), respectively a \( \gamma \)-radonifying operator (in part (b)), denoted by \( R_T \), from \( \mathcal{H}_T \) to \( E \). We estimate
\[
\left\| \sum_{n \geq 1} \alpha_n h_n \right\|_{\mathcal{H}_T} = \left\| \sum_{n \geq 1} \alpha_n Th_n \right\|_{\mathcal{H}} \leq C_B^{-1} \left\| \sum_{n \geq 1} \alpha_n h_n \right\|_{\mathcal{H}} = C_B^{-1} \left( \sum_{n \geq 1} |\alpha_n|^2 \right)^{1/2}.
\]
From this we deduce that \( (jh_n)_{n \geq 1} \) is a Hilbert sequence in \( \mathcal{H}_T \) with constant \( \leq C_B^{-1} \). Hence we may apply Proposition 2.1 to the operator \( R_T : \mathcal{H}_T \to E \) and the Hilbert sequence \( (jh_n)_{n \geq 1} \) in \( \mathcal{H}_T \) to obtain the result.

\[\square\]
As a consequence of the above results we obtain:

**Theorem 2.3.** Let \((f_n)_{n \geq 1}\) be a Riesz basis in the Hilbert space \(\mathcal{H}\).

(a) One has \(R \in \gamma_\infty(\mathcal{H}, E)\) if and only if \(\sup_{N \geq 1} \left\| \sum_{n=1}^{N} \gamma_n R f_n \right\|_{L^2(\Omega, E)} < \infty\). In that case (1) and (3) hold.

(b) One has \(R \in \gamma(\mathcal{H}, E)\) if and only if \(\sum_{n \geq 1} \gamma_n R f_n\) converges in \(L^2(\Omega, E)\). In that case (2) and (4) hold.

The following well-known lemma identifies a class of Riesz sequences in \(L^2(\mathbb{R})\). For convenience we include the short proof from [1, Theorem 2.1]. Let \(\mathbb{T}\) be the unit circle in \(\mathbb{C}\).

**Lemma 2.4.** Let \(f \in L^2(\mathbb{R})\) and define the sequence \((f_n)_{n \in \mathbb{Z}}\) in \(L^2(\mathbb{R})\) by \(f_n(t) = e^{2\pi i n t} f(t)\). Define \(F : \mathbb{T} \to \mathbb{R}\) as

\[
F(e^{2\pi i t}) := \sum_{k \in \mathbb{Z}} |f(t + k)|^2
\]

(a) The sequence \((f_n)_{n \in \mathbb{Z}}\) is a Bessel sequence in \(L^2(\mathbb{R})\) if and only if there exists a constant \(A > 0\) such that \(A \leq F(e^{2\pi i t})\) for almost all \(t \in [0, 1]\).

(b) The sequence \((f_n)_{n \in \mathbb{Z}}\) is a Hilbert sequence in \(L^2(\mathbb{R})\) if and only if there exists a constant \(B > 0\) such that \(F(e^{2\pi i t}) \leq B\) for almost all \(t \in [0, 1]\).

In these cases, \(C_B^2 = \text{ess inf} F\) and \(C_B^2 = \text{ess sup} F\) respectively.

**Proof.** Both assertions are obtained by observing that for \(I \subseteq \mathbb{Z}\) and \((a_n)_{n \in I}\) in \(\mathbb{C}\) we may write

\[
\left\| \sum_{n \in I} a_n f_n \right\|_{L^2(\mathbb{R})}^2 = \sum_{k \in \mathbb{Z}} \int_{I} \left| \sum_{n \in I} a_n e^{2\pi i n t} f(t) \right|^2 \, dt
\]

\[
= \sum_{k \in \mathbb{Z}} \int_{0}^{1} \left| \sum_{n \in I} a_n e^{2\pi i n t} f(t + k) \right|^2 \, dt = \int_{0}^{1} \left| \sum_{n \in I} a_n e^{2\pi i n t} \right|^2 F(e^{2\pi i t}) \, dt.
\]

The following application of Lemma 2.4 will be used below.

**Example 2.5.** Let \(\rho \in [0, 1)\) and \(a > 0\). For \(n \in \mathbb{Z}\) let

\[
f_n(t) = e^{-a t + 2\pi(n + \rho) i t} I_{[0,\infty)}(t).
\]

Then \((f_n)_{n \in \mathbb{Z}}\) is a Riesz sequence in \(L^2(\mathbb{R})\) with constants \(C_B^2 = \frac{e^{-2a}}{e^{2a} - 1}\) and \(C_B^2 = \frac{e^{2a}}{e^{2a} - 1}\). Indeed, let \(f(t) := e^{-a t + 2\pi i \rho t} I_{[0,\infty)}(t)\). For all \(t \in [0, 1)\),

\[
F(e^{2\pi i t}) = \sum_{k \in \mathbb{Z}} |f(t + k)|^2 = \sum_{k=0}^{\infty} e^{-2a(t+k)} = \frac{e^{2a(1-t)}}{e^{2a} - 1}.
\]

Now Lemma 2.4 implies the result.

**Remark 2.6.** Necessary and sufficient conditions on the complex coefficients \(c_n\) and \(\lambda_n\) with \(\text{Re} \lambda_n > 0\) in order that the functions \(z \mapsto c_n \exp(-\lambda_n z)\) form a Riesz sequence can be found in [13, Section 10.3] and [7].
3 Main results

In this section we use Proposition 2.1 to obtain an alternative proof of [12, Theorem 3.4] on the $R$-boundedness of certain Laplace transforms. This result is applied to strongly continuous semigroups to obtain estimates for the abscissa of $R$-boundedness of the resolvent. From this we deduce Theorem 1.1a as well as bounded perturbation results for the existence of solutions and invariant measures for the problem $(SCP)_{(A,B)}$.

Let $(r_n)_{n \geq 1}$ be a Rademacher sequence on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A family of operators $\mathcal{F} \subseteq \mathcal{B}(E)$ is called $R$-bounded if there exists a constant $C > 0$ such that for all $N \geq 1$ and all sequences $(T_n)_{n=1}^N \subseteq \mathcal{F}$ and $(x_n)_{n=1}^N \subseteq E$ we have

$$E \left\| \sum_{n=1}^N r_n T_n x_n \right\|^2 \leq C^2 E \left\| \sum_{n=1}^N r_n x_n \right\|^2.$$

The least possible constant $C$ is called the $R$-bound of $\mathcal{F}$, notation $\mathcal{A}(\mathcal{F})$. Clearly, every $R$-bounded family $\mathcal{F}$ is uniformly bounded and $\sup_{T \in \mathcal{F}} \|T\| \leq \mathcal{A}(\mathcal{F})$.

Following [12], for an operator $T \in \mathcal{B}(L^2(\mathbb{R}_+), E)$ we define the Laplace transform $\hat{T} : \{ \lambda \in \mathbb{C} : \text{Re}\lambda > 0 \} \to E$ as

$$\hat{T}(\lambda) := Te^\lambda.$$

Here $e_\lambda \in L^2(\mathbb{R}_+)$ is given by $e_\lambda(t) = e^{-\lambda t}$. For a Banach space $F$ and a bounded operator $\Theta : F \to \mathcal{B}(L^2(\mathbb{R}_+), E)$ we define the Laplace transform $\hat{\Theta} : \{ \lambda \in \mathbb{C} : \text{Re}\lambda > 0 \} \to \mathcal{B}(F,E)$ as

$$\hat{\Theta}(\lambda)y := \hat{\Theta}y(\lambda) \quad \text{Re}\lambda > 0, \; y \in F.$$

The following result is a slight refinement of [12, Theorem 3.4]. The main novelty is the simple proof of the estimate (5).

**Theorem 3.1.** Let $F$ be a Banach space. Let $\Theta : F \to \gamma_\infty(L^2(\mathbb{R}_+), E)$ be a bounded operator and let $\delta > 0$. Then $\hat{\Theta}$ is $R$-bounded on the half-plane $\{ \lambda \in \mathbb{C} : \text{Re}\lambda > \delta \}$ and there exists a universal constant $C$ such that

$$\mathcal{R} \left( \{ \hat{\Theta}(\lambda) : \text{Re}\lambda \geq \delta \} \right) \leq \|\Theta\| \frac{C}{\sqrt{\delta}}.$$

**Proof.** Let $\delta > 0$. Consider the set $\{ \lambda \in \mathbb{C} : \text{Re}\lambda = \delta \}$. Fix $\sigma \in [\delta/2, 3\delta/2]$ and $\rho \in [0,1)$. For $n \in \mathbb{Z}$ let $g_n : \mathbb{R} \to \mathbb{C}$ be given by $g_n(t) = e^{-\sigma t + (n+\rho)\delta t}$.

By a substitution, this reduces to Example 2.5, whence $(g_n)_{n \geq 1}$ is a Riesz sequence in $L^2(\mathbb{R}_+)$ with constant $0 < C_H \leq \left( \frac{C}{\delta} \right)^{\delta/2}$ where $C := 2\pi \frac{e^{\pi^2/4}}{e^{-\pi^2/4}}$. For $y \in F$, we may apply Proposition 2.1 to obtain

$$\left\| \sum_{n=-N}^N \gamma_n \hat{\Theta}(\sigma - (n+\rho)\delta i)y \right\|_{L^2(\Omega,E)} \leq C_H \|\Theta y\|_{\gamma_{\infty}(\Omega,E)} \leq \left( \frac{C}{\delta} \right)^{\delta/2} \||\Theta|| \|y\|.$$

(5)

The rest of the proof follows the lines in [12].
In what follows we let \((T(t))_{t \geq 0}\) be a strongly continuous semigroup on \(E\) with generator \(A\). We recall from [11, 12] that the problem (SCP)\(_{(A,B)}\) admits a (unique) solution if and only if \(T(\cdot)B\) belongs to \(\gamma([0,T],H,E)\) for some (all) \(T > 0\). Furthermore, an invariant measure exists if and only if \(T(\cdot)B\) belongs to \(\gamma(\mathbb{R}_+,H,E)\).

The next theorem improves [12, Theorem 1.3], where the bound \(s_R(A) \leq 0\) was obtained.

**Theorem 3.2.** Assume that for all \(x \in E\), \(T(\cdot)x \in \gamma_\infty(\mathbb{R}_+,E)\). Then \(s_R(A) < 0\), i.e., there exists an \(\varepsilon > 0\) such that \(\{R(\lambda,A) : \Re\lambda \geq -\varepsilon\} \) is \(R\)-bounded.

**Proof.** By the closed graph theorem there exists an \(\varepsilon > 0\) such that \(\|T(\cdot)x\|_{\gamma_\infty(\mathbb{R}_+,E)} \leq M\|x\|\). By Theorem 3.1, \(\{\lambda \in \mathbb{C} : \Re\lambda > 0\} \subseteq \phi(A)\) and

\[
\mathcal{R}(\{R(\lambda,A) : \Re\lambda \geq \delta\}) \leq \frac{c}{\sqrt{\delta}}
\]

for all \(\delta > 0\), where \(c := CM\) with \(C\) the universal constant of Theorem 3.1. The following standard argument shows that this implies the bound

\[
s(A) \leq -\frac{1}{4c^2}.
\]  

Choose \(\delta > 0\) and let \(\mu \in \sigma(A)\) be such that \(\Re\mu > s(A) - \delta\). With \(\lambda = \frac{1}{4c^2} + i\Im\mu\) it follows that

\[
\frac{1}{4c^2} - s(A) + \delta \geq \text{dist}(\lambda, \sigma(A)) \geq \frac{1}{\|R(\lambda,A)\|} \geq \frac{\sqrt{\Re\lambda}}{c} = \frac{1}{2c^2}.
\]

Thus \(s(A) \leq -\frac{1}{4c^2} + \delta\). Since \(\delta > 0\) was arbitrary, this gives (7).

Now let \(\varepsilon_0 := \frac{1}{4c^2}\). For \(\lambda\) with \(-\varepsilon_0 < \Re\lambda < 3\varepsilon_0\) we may write

\[
R(\lambda,A) = \sum_{n \geq 0} (\varepsilon_0 - \Re\lambda)^n R(\varepsilon_0 + i\Im\lambda, A)^{n+1}.
\]

Fix \(0 < \varepsilon < \varepsilon_0\). We claim that \(\{R(\lambda,A) : \Re\lambda = -\varepsilon\} \) is \(R\)-bounded. To see this let \((r_k)_{k=1}^K\) be a Rademacher sequence on \((\Omega,\mathcal{F},\mathbb{P})\), let \((\lambda_k)_{k=1}^K\) be such that \(\Re\lambda_k = -\varepsilon\), and let \((x_k)_{k=1}^K\) be a sequence in \(E\). We may estimate

\[
\left\| \sum_{k=1}^K r_k R(\lambda_k,A)x_k \right\|_{L^2(\Omega,E)} = \left\| \sum_{n \geq 0} \sum_{k=1}^K r_k (\varepsilon_0 + \varepsilon)^n R(\varepsilon_0 + i\Im\lambda_k, A)^{n+1} x_k \right\|_{L^2(\Omega,E)}
\]

\[
\leq \sum_{n \geq 0} (\varepsilon_0 + \varepsilon)^n \left\| \sum_{k=1}^K r_k R(\varepsilon_0 + i\Im\lambda_k, A)^{n+1} x_k \right\|_{L^2(\Omega,E)}
\]

\[
\leq \sum_{n \geq 0} (\varepsilon_0 + \varepsilon)^n \left( \frac{c}{\varepsilon_0} \right)^{n+1} \left\| \sum_{k=1}^K r_k x_k \right\|_{L^2(\Omega,E)}
\]

\[
= \frac{1}{\varepsilon_0 - \varepsilon} \left\| \sum_{k=1}^K r_k x_k \right\|_{L^2(\Omega,E)}
\]

where we used that \(\varepsilon_0 = \frac{1}{4c^2}\). This proves the claim. Now the result is obtained via [16, Proposition 2.8].
As an application of Theorem 3.2 we have the following bounded perturbation result for the existence of a solution for the perturbed problem.

**Theorem 3.3.** Let $P \in \mathcal{B}(E)$ and $B \in \mathcal{B}(H, E)$. If $(SCP)_{(A,B)}$ has a solution, then $(SCP)_{(A+P,B)}$ has a solution as well.

**Proof.** For $\omega \in \mathbb{R}$ denote $A_\omega = A - \omega$ and $T_\omega (\cdot) := e^{-\omega T(\cdot)}$. It follows from [12, Proposition 4.5] that for all $\omega > \omega_0(A)$, $T_\omega (\cdot) B \in \gamma(\mathbb{R}_+, H, E)$. From [9, Corollary 2.17] it follows that for all $\omega > \omega_0(A) + 1$, $$\mathcal{R} (\{R(\lambda, A_\omega) : \text{Re} \lambda \geq 0\}) \leq \frac{c}{\omega - \omega_0(A) - 1},$$ where $c$ is a constant depending only on $T$. Choose $\omega_1 > \omega_0(A) + 1$ so large that $\frac{c}{\omega_1 - \omega_0(A) - 1} \|P\| < 1$. By [12, Lemma 5.1], $R(i_\omega, A_\omega) B \in \gamma(\mathbb{R}_+, H, E)$.

Denote by $(S(t))_{t \geq 0}$ the semigroup generated by $A + P$ (cf. [5, Section III.1] or [15, Chapter III]) and let $S_\omega (\cdot) := e^{-\omega t S(t)}$, $t \geq 0$. Since $$\mathcal{R} (\{R(is, A_\omega) : s \in \mathbb{R}\}) \leq \mathcal{R} (\{R(is, A_\omega) : s \in \mathbb{R}\}) \|P\| =: C < 1,$$ it follows from $i \mathbb{R} \subseteq \sigma(A_\omega)$ that $i \mathbb{R} \subseteq \sigma(A_\omega + P)$ and $$R(is, A_\omega + P) B = \sum_{n=0}^{\infty} (R(is, A_\omega) P)^n R(is, A_\omega) B =: R_{A,P_\omega}(s) R(is, A_\omega) B.$$ Moreover, as in Theorem 3.2, and using the fact that $C < 1$, $\{R_{A,P_\omega}(s) : s \in \mathbb{R}\}$ is $R$-bounded with constant $\frac{1}{1-C}$. From [8, Proposition 4.11] we deduce that $$\|R(i_\omega, A_\omega + P) B\|_{\gamma(\mathbb{R}, H, E)} \leq \frac{1}{1-C} \|R(i_\omega, A_\omega) B\|_{\gamma(\mathbb{R}, H, E)}.$$ Now [12, Lemma 5.1] shows that $S_\omega (\cdot) B \in \gamma(\mathbb{R}_+, H, E)$. It follows from the right ideal property that for all $t > 0$, $$\|S(\cdot) B\|_{\gamma(0,t, H, E)} \leq e^{\omega t} \|S_\omega (\cdot) B\|_{\gamma(0,1, H, E)}$$ and the result can be obtained via [11, Theorem 7.1]. \(\square\)

Concerning existence and uniqueness of invariant measures we obtain:

**Theorem 3.4.** Assume that $s(A) < 0$ and that $\{R(is, A) : s \in \mathbb{R}\}$ is $R$-bounded. Let $B \in \mathcal{B}(H, E)$ such that $(SCP)_{(A,B)}$ admits an invariant measure. Then there exists a $\delta > 0$ such that for all $P \in \mathcal{B}(E)$ with $\|P\| < \delta$, $(SCP)_{(A+P,B)}$ admits a unique invariant measure.

**Proof.** Let $\delta > 0$ such that $\mathcal{R} (\{R(is, A) : s \in \mathbb{R}\}) \leq \frac{1}{\delta}$. Then, if $\|P\| < \delta$, $$\mathcal{R} (\{R(is, A) P : s \in \mathbb{R}\}) \leq \mathcal{R} (\{R(is, A) : s \in \mathbb{R}\}) \|P\| =: C < 1.$$ As in Theorem 3.3 it can be deduced that $$\|R(i_\omega, A + P) B\|_{\gamma(\mathbb{R}, H, E)} \leq \frac{1}{1-C} \|R(i_\omega, A) B\|_{\gamma(\mathbb{R}, H, E)}.$$ The existence of an invariant measure now follows from [12, Proposition 4.4 and Lemma 5.1].

By [12, Corollary 4.3], for uniqueness it suffices to note that $R(\lambda, A + P)$ is uniformly bounded for $\text{Re} \lambda > 0$. \(\square\)
In particular, the $R$-boundedness of $\{R(is, A) : s \in \mathbb{R}\}$ implies that an invariant measure for $(SCP)_{(A,B)}$ if one exists, is unique. On the other hand, if $i\mathbb{R} \subseteq \rho(A)$ but $\{R(is, A) : s \in \mathbb{R}\}$ fails to be $R$-bounded, then Theorem 3.2 shows that there exists a rank one operator $B' \in \mathcal{B}(H, E)$ such that the problem $(SCP)_{(A,B')}$ fails to have an invariant measure. As a result we obtain that if $(SCP)_{(A,B)}$ fails to have a unique invariant measure, then there exists a rank one operator $B' \in \mathcal{B}(H, E)$ such that the problem $(SCP)_{(A,B')}$ fails to have an invariant measure. A related result can be found in [6].

**Proof of Theorems 1.1a and 1.1b.** If $T(\cdot)x \in \gamma(\mathbb{R}_+, E)$ for all $x \in E$, then by Theorem 3.2 $s(A) < 0$ and $\{R(is, A) : s \in \mathbb{R}\}$ is $R$-bounded. Thus, Theorem 3.4 applies to the bounded perturbation $P = \delta \cdot I_E$. 

**References**


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