$B-$convexity, the analytic Radon-Nikodym property, and individual stability of $C_0$-semigroups

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and

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Let $T = \{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup on a Banach space $X$, with generator $A$ and growth bound $\omega$. Assume that $x_0 \in X$ is such that the local resolvent $\lambda \mapsto R(\lambda, A)x_0$ admits a bounded holomorphic extension to the right half-plane $\{ \Re \lambda > 0 \}$. We prove the following results:

(i) If $X$ has Fourier type $p \in (1, 2]$, then $\lim_{t \to \infty} \|T(t)(\lambda_0 - A)^{-\beta}x_0\| = 0$ for all $\beta > \frac{1}{p}$ and $\lambda_0 > \omega$.

(ii) If $X$ has the analytic RNP, then $\lim_{t \to \infty} \|T(t)(\lambda_0 - A)^{-\beta}x_0\| = 0$ for all $\beta > 1$ and $\lambda_0 > \omega$.

(iii) If $X$ is arbitrary, then weak-$\lim_{t \to \infty} T(t)(\lambda_0 - A)^{-\beta}x_0 = 0$ for all $\beta > 1$ and $\lambda_0 > \omega$.

As an application we prove a Tauberian theorem for the Laplace transform of functions with values in a $B-$convex Banach space.

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0. Introduction

In this paper we address the problem to find sufficient conditions on the local spectra of individual orbits of a $C_0$-semigroup $T = \{T(t)\}_{t \geq 0}$ to ensure their strong convergence to zero. In recent work [1,2,9,15] it has become increasingly clear that most of the ‘global’ stability theory can be localized to individual orbits $T(\cdot)x$ by replacing the assumptions on the spectrum of the generator $A$ to assumptions of the local spectrum of $A$ at $x$.

For example, it has been proved by Weis and Wrobel [22] that $T$ is exponentially stable, i.e. there exist $M > 0$ and $\omega > 0$ such that $\|T(t)x\| \leq Me^{-\omega t}\|x\|_{D(A)}$ for all $x \in D(A)$, if the resolvent $R(\lambda, A) = (\lambda - A)^{-1}$ exists and is uniformly bounded in the right half-plane $\{\Re \lambda > 0\}$. A little later and independently, in [15] the following local version of this result was proved: if $x_0 \in X$ is such that the map $\lambda \mapsto R(\lambda, A)x_0$ admits a bounded holomorphic extension to $\{\Re \lambda > 0\}$, then for each $\lambda_0 \in \rho(A)$ there exists a constant $M > 0$ such that

$$\|T(t)R(\lambda_0, A)x_0\| \leq M(1 + t), \quad t \geq 0.$$  

By a standard resolvent expansion argument, the Weis-Wrobel result is an immediate consequence of this. In [9], for Hilbert spaces it was proved that actually

$$\lim_{t \to \infty} \|T(t)R(\lambda_0, A)x_0\| = 0.$$  

In this paper, we extend the result of [9] into various directions.

Let $p \in [1, 2]$. A Banach space $X$ has Fourier type $p$ if the Fourier transform extends to a bounded linear operator from $L^p(\mathbb{R}, X)$ into $L^q(\mathbb{R}, X)$, $\frac{1}{p} + \frac{1}{q} = 1$. Trivially, every Banach space has Fourier type $p = 1$, but certain spaces have non-trivial Fourier type; see Section 1.

A Banach space $X$ has the analytic Radon-Nikodym property if for every $f \in H^p(D, X)$, the Hardy space of all $X$-valued holomorphic functions on the unit disc $D$, the radial limits $\lim_{r \to 1} f(re^{i\theta})$ exist for almost all $\theta \in [0, 2\pi]$. This property will be discussed in more detail in Section 2.

Our main results read as follows.

**Theorem 0.1.** Let $X$ be a Banach space with Fourier type $p \in (1, 2]$ and let $A$ be the generator of a $C_0$-semigroup $T$ on $X$. If $x_0 \in X$ is such that the map $\lambda \mapsto R(\lambda, A)x_0$ admits a bounded holomorphic extension in the open right half-plane, then for all $\beta > \frac{1}{p}$ and $\lambda_0 > \omega_0(T)$ we have

$$\lim_{t \to \infty} \|T(t)(\lambda_0 - A)^{-\beta}x_0\| = 0.$$
Theorem 0.2. Let \( X \) be a Banach space with the analytic Radon-Nikodym property and let \( A \) be the generator of a \( C_0 \)-semigroup \( T \) on \( X \). If \( x_0 \in X \) is such that the map \( \lambda \mapsto R(\lambda, A)x_0 \) admits a bounded holomorphic extension in the open right half-plane, then for all \( \beta > 1 \) and \( \lambda_0 > \omega_0(T) \) we have
\[
\lim_{t \to \infty} \|T(t)(\lambda_0 - A)^{-\beta}x_0\| = 0.
\]

Theorem 0.3. Let \( A \) be the generator of a \( C_0 \)-semigroup \( T \) on an arbitrary Banach space \( X \). If \( x_0 \in X \) is such that the map \( \lambda \mapsto R(\lambda, A)x_0 \) admits a bounded holomorphic extension in the open right half-plane, then for all \( \beta > 1 \) and \( \lambda_0 > \omega_0(T) \) we have
\[
\lim_{t \to \infty} T(t)(\lambda_0 - A)^{-\beta}x_0 = 0.
\]

In these results, \( \omega_0(T) \) denotes the growth bound of \( T \), i.e. the infimum of all \( \omega \in \mathbb{R} \) such that \( \|T(t)\| \leq Me^{\omega t} \) for some \( M > 0 \) and all \( t \geq 0 \). The restriction to real \( \lambda_0 \) is not essential; by a standard rescaling argument the same results hold for \( \lambda_0 \in \mathbb{C} \) with \( \Re \lambda_0 > \omega_0(T) \).

We also present a simple example which shows that \( \lim_{t \to \infty} \|T(t)(\lambda_0 - A)^{-\beta}x_0\| = 0 \) may fail for all \( \beta \geq 0 \) if no restrictions on the Banach space \( X \) are imposed.

The paper is organized as follows. In Section 1 we prove Theorems 0.1 and 0.3 and give a simple application to Tauberian theory for the Laplace transform of functions with values in a Banach space with non-trivial Fourier type. In Section 2 we present the proof of Theorem 0.2 and a second proof of Theorem 0.3.

1. Stability and \( B \)-convexity

Let \( A \) be a closed, densely defined operator in a Banach space \( X \) such that \((0, \infty) \subset \rho(A)\), the resolvent set of \( A \), and assume that there is a constant \( M > 0 \) such that
\[
\|R(\lambda, A)\| \leq \frac{M}{1 + \lambda}, \quad \lambda > 0.
\]  

As is well-known, fractional powers of \(-A\) can be defined, and for \( 0 < \beta < 1 \) we have the representation
\[
(-A)^{-\beta}x = \frac{\sin \pi \beta}{\pi} \int_0^\infty t^{-\beta}R(t, A)x \, dt, \quad x \in X.
\]  

For the theory of fractional powers the reader is referred to [21].
If \( A \) is the generator of a \( C_0 \)-semigroup \( T \), then for all \( \lambda_0 > \omega_0(T) \) the operator \( A - \lambda_0 \) satisfies an estimate of the type (1.1), and the fractional powers of \( \lambda_0 - A \) are well-defined. We assume that the reader is familiar with the elementary theory of \( C_0 \)-semigroups; we refer to [14,17].

Let \( p \in [1,2] \). A Banach space \( Y \) has \textit{Fourier type} \( p \) if the \( Y \)-valued Hausdorff-Young theorem holds, i.e. if the Fourier transform extends to a bounded linear operator from \( L^p(\mathbb{R},Y) \) into \( L^q(\mathbb{R},Y) \), \( \frac{1}{p} + \frac{1}{q} = 1 \). Here, as usual, for \( f \in L^p(\mathbb{R},Y) \cap L^1(\mathbb{R},Y) \), the Fourier transform \( \mathcal{F}f \) is defined by

\[
\mathcal{F}f(s) := \int_{-\infty}^{\infty} e^{-ist} f(t) \, dt, \quad s \in \mathbb{R}.
\]

Every Banach space has Fourier type 1 but only Banach spaces which are isomorphic to Hilbert spaces have Fourier type 2 [12]. The classical spaces \( L^p(\mu) \) have Fourier type \( \min\{p,q\} \), \( \frac{1}{p} + \frac{1}{q} = 1 \) [18].

A Banach space \( Y \) is called \textit{B-convex} if \( Y \) does not contain the spaces \( l^n_1 \) uniformly, or equivalently, if it has non-trivial type, i.e. if it has type \( p \) for some \( p \in (1,2] \). The spaces \( L^p(\mu) \) are B-convex and more generally, every Lebesgue-Bochner space \( L^p(\mu,Y) \) with \( Y \) B-convex is B-convex (cf. [13, p. 247]) and every uniformly convex Banach space is B-convex. For more details the reader should consult [19]. Every B-convex Banach space has non-trivial Fourier type , i.e. Fourier type \( p \) for some \( p \in (1,2] \) [4], and conversely it is easy to show that a space with non-trivial Fourier type is B-convex (cf. [3, p. 354]).

In most of the results of this section, we investigate the behaviour of the map \( t \mapsto PT(t)(\lambda_0 - A)^{-\beta}x_0 \), assuming certain growth conditions on \( \lambda \mapsto PR(\lambda,A)x_0 \); here, \( P \) is an arbitrary bounded linear operator from \( X \) into some B-convex Banach space \( Y \). Although we are primarily interested in the case \( Y = X \) and \( P = I \), this slightly more general setting allows the following applications:

- Taking \( Y = \mathbb{C} \) and \( P = x^* \in X^* \) we obtain weak analogues of our results;
- We may consider the translation semigroup on \( X = BUC(\mathbb{R}_+,Y) \) and the map \( P : X \to Y, Pf := f(0) \). In this way the asymptotic behaviour of \( Y \)-valued \( BUC \)-functions can be studied via semigroup techniques;
- It may be possible to apply our results to matrix semigroups, taking for \( P \) a coordinate projection. Matrix semigroups arise, e.g., in the study of delay equations and higher order abstract Cauchy problems.

The first lemma imposes no restrictions on the Fourier type of \( Y \).

\textbf{Lemma 1.1.} \textit{Let} \( X \) \textit{and} \( Y \) \textit{be Banach spaces and let} \( P : X \to Y \) \textit{be a bounded linear operator}. \textit{Let} \( A \) \textit{be the generator of a} \( C_0 \)-semigroup \( T \) \textit{on} \( X \) \textit{and let} \( x_0 \in X \) \textit{be}
such that the map \( \lambda \mapsto PR(\lambda, A)x_0 \) admits a holomorphic extension \( F(\lambda) \) in the open right half-plane. Suppose there exist \( \omega_0 > \max\{0, \omega_0(T)\} \), \( M > 0 \), and \( \alpha \in [-1, \infty) \) such that

\[
\|F(\lambda)\| \leq M(1 + |\lambda|)^\alpha, \quad 0 < \Re \lambda < \omega_0.
\]

Fix \( \lambda_0 > \max\{0, \omega_0(T)\} \). For all \( \beta \geq 0 \) with \( \beta > \alpha \) the function \( \lambda \mapsto PR(\lambda, A)(\lambda_0 - A)^{-\beta}x_0 \) (\( \Re \lambda > \omega_0(T) \)) admits a holomorphic extension \( g(\lambda) \) in the open right half-plane, and for all \( \omega_1 \in (0, \min\{\omega_0, \lambda_0\}) \) there exists a constant \( C > 0 \) such that

\[
\|g(\lambda)\| \leq C(1 + |\lambda|)^{\max\{\alpha - \beta, -1\}}, \quad 0 < \Re \lambda < \omega_1.
\] (1.3)

Proof: Fix \( \lambda_0 > \max\{0, \omega_0(T)\} \) and \( 0 < \omega_1 < \min\{\omega_0, \lambda_0\} \). Upon replacing \( \omega_0 \) by some smaller number and \( \omega_1 \) by a larger, we may assume that \( \max\{0, \omega_0(T)\} < \omega_1 < \omega_0 < \lambda_0 \).

Let \( \beta = n + \delta \) with \( n \in \mathbb{N} \) and \( 0 \leq \delta < 1 \) and put \( y_0 := R(\lambda_0, A)^nx_0 \). In view of the identity

\[
R(\lambda, A)y_0 = \frac{R(\lambda, A)x_0}{(\lambda_0 - \lambda)^n} - \sum_{k=0}^{n-1} \frac{R(\lambda_0, A)^{k+1}x_0}{(\lambda_0 - \lambda)^{n-k}},
\]

the map \( \lambda \mapsto PR(\lambda, A)y_0 \) admits a holomorphic extension \( F_1(\lambda) \) to \( \{\Re \lambda > 0\} \) which satisfies

\[
\|F_1(\lambda)\| \leq M'(1 + |\lambda|)^{\max\{\alpha - n, -1\}}, \quad 0 < \Re \lambda < \omega_0,
\] (1.4)

for some constant \( M' > 0 \).

If \( \delta = 0 \) (so \( \beta = n \)), then \( g = F_1 \) and the proof is complete. Therefore, in the rest of the proof we will assume that \( \delta \in (0, 1) \).

We have

\[
g(\lambda) = PR(\lambda, A)(\lambda_0 - A)^{-\beta}x_0 = PR(\lambda, A)(\lambda_0 - A)^{-\delta}y_0, \quad \Re \lambda > \omega_0(T).
\]

Hence by (1.2) and the resolvent identity, for \( \Re \lambda > \omega_0(T) \) we have

\[
g(\lambda) = \frac{\sin \pi \delta}{\pi} \int_0^\infty t^{-\delta} PR(\lambda, A)R(\lambda_0 + t, A)y_0 \, dt
\]

\[
= \frac{\sin \pi \delta}{\pi} \int_0^\infty t^{-\delta} \frac{PR(\lambda, A)y_0 - PR(\lambda_0 + t, A)y_0}{t + \lambda_0 - \lambda} \, dt.
\]

Passing to the holomorphic extension, we see that

\[
g(\lambda) = \frac{\sin \pi \delta}{\pi} \int_0^\infty t^{-\delta} \frac{F_1(\lambda) - F_1(\lambda_0 + t)}{t + \lambda_0 - \lambda} \, dt;
\] (1.5)
by (1.4) and the fact that $\alpha < \beta = n + \delta$ this integral converges absolutely and defines a holomorphic extension of $g$ in the strip $\{0 < \Re \lambda < \lambda_0\}$.

For $\omega > 0$ consider the functions $g_\omega : \mathbb{R} \to Y$ defined by

$$g_\omega(s) := g(\omega - is), \quad s \in \mathbb{R}.$$ 

Then $g_\omega(s) = PR(\omega - is, A)(\lambda_0 - A)^{-\beta} x_0$ for $\omega > \omega_0(T)$. Noting that $\|R(\lambda, A)\| \leq \text{const} \cdot (\Re \lambda - \omega_0)^{-1}$ for all $\Re \lambda > \lambda_0$, we see that $c := \sup_{\tau \geq \lambda_0} \tau \|F_1(\tau)\| < \infty$. Hence by (1.4) and (1.5), for all $0 < \omega < \omega_1$ and $s \in \mathbb{R}$ we have

$$\|g_\omega(s)\| \leq \frac{\sin \pi \delta}{\pi} \int_0^\infty t^{-\frac{\delta}{2}} e^{(1 + (\omega^2 + s^2)^{\frac{\delta}{2}})^{\max\{\alpha - n, -1\}}} + c(\lambda_0 + t)^{-1} dt$$

$$\leq \text{const} \cdot (1 + s^2)^{\frac{\delta}{2} \max\{\alpha - n - \delta, -1\}},$$

where the constant is independent of $s \in \mathbb{R}$ and $\omega \in (0, \omega_1)$. 

We can now state and prove the first main result.

**Theorem 1.2.** Let $P$ be a bounded linear operator from a Banach space $X$ into a Banach space $Y$ with Fourier type $p \in (1, 2)$. Let $A$ be the generator of a $C_0$-semigroup $T$ on $X$ and let $x_0 \in X$ be such that the map $\lambda \mapsto PR(\lambda, A)x_0$ admits a holomorphic extension $F(\lambda)$ in the open right half-plane. If there exist $\omega_0 > \max\{0, \omega_0(T)\}$, $M > 0$, and $\alpha \in [-1, \infty)$ such that

$$\|F(\lambda)\| \leq M(1 + |\lambda|)^{\alpha}, \quad 0 < \Re \lambda < \omega_0,$$

then for all $\beta \geq 0$ with $\beta > \alpha + \frac{1}{p}$ and all $\lambda_0 > \omega_0(T)$ we have

$$PT(\cdot)(\lambda_0 - A)^{-\beta} x_0 \in L^q(\mathbb{R}_+, Y), \quad \frac{1}{p} + \frac{1}{q} = 1.$$

**Proof:** Without loss of generality we may assume that $\omega_0(T) \geq 0$. Fix $\lambda_0 > \omega_0(T)$. By taking a smaller value of $\omega_0$, we may furthermore assume that $\omega_0(T) < \omega_0 < \lambda_0$. Fix $\omega_1 \in (\omega_0(T), \omega_0)$.

Let the functions $g_\omega$ be defined as in the proof of Lemma 1.1. In view of $\beta - \alpha > \frac{1}{p}$ and $p > 1$ the estimate obtained there shows that $g_\omega \in L^p(\mathbb{R}, Y)$, uniformly for $\omega \in (0, \omega_1)$. Let $C := \sup_{0 < \omega < \omega_1} \|g_\omega\|_p$.

Since $Y$ has Fourier type $p$, the Fourier transform $G_\omega := \frac{1}{2\pi} \mathcal{F} g_\omega$ of $g_\omega$ defines an element of $L^q(\mathbb{R}, Y)$.

Let $\omega \in (0, \omega_1)$ be fixed. We claim that

$$G_\omega(t) = e^{-\omega t} PT(t)(\lambda_0 - A)^{-\beta} x_0 \quad \text{for a.a. } t > 0.$$
To see this we define, for each \( r > 0 \), \( g_{\omega, r} := g_\omega \cdot \chi_{[-r, r]} \). Then \( \lim_{r \to \infty} g_{\omega, r} = g_\omega \) in the norm of \( L^p(\mathbb{R}, Y) \), so for the Fourier transforms \( G_{\omega, r} = \frac{1}{2\pi} \mathcal{F}g_{\omega, r} \) we have \( \lim_{r \to \infty} G_{\omega, r} = G_\omega \) in \( L^q(\mathbb{R}, Y) \). Let \( \Gamma \) be the rectangle spanned by the points \( \omega - ir, \omega + ir, \omega_0 + ir, \) and \( \omega_0 - ir \). By Cauchy’s theorem, for all \( t > 0 \) we have

\[
\frac{1}{2\pi i} \int_{\omega - ir}^{\omega + ir} e^{zt} g(z) \, dz = \frac{1}{2\pi i} \int_{\omega_0 - ir}^{\omega_0 + ir} e^{zt} g(z) \, dz + R_r(t) \\
= \frac{1}{2\pi i} \int_{\omega_0 - ir}^{\omega_0 + ir} e^{zt} PR(z, A)(\lambda_0 - A)^{-\beta} x_0 \, dz + R_r(t),
\]

(1.6)

where \( R_r(t) \) represents the integrals over the two horizontal parts of \( \Gamma \). From (1.3) we see that \( \lim_{r \to \infty} \| R_r(t) \| = 0 \) for all \( t > 0 \). Also, by the complex inversion theorem for the Laplace transform, the Cesàro means of the integral on the right hand side in (1.6) converge to \( PT(t)(\lambda_0 - A)^{-\beta} x_0 \) as \( r \to \infty \); here we use that \( \omega_0 > \omega_0(T) \). It follows that for all \( t > 0 \),

\[
\lim_{m \to \infty} \frac{1}{m} \int_0^m \frac{1}{2\pi i} \int_{\omega - ir}^{\omega + ir} e^{zt} g(z) \, dz \, dr = PT(t)(\lambda_0 - A)^{-\beta} x_0.
\]

(1.7)

On the other hand, for \( t > 0 \) we have

\[
G_{\omega, r}(t) = \frac{1}{2\pi} \int_{-r}^r e^{-ist} g(\omega - is) \, ds = \frac{1}{2\pi i} e^{-\omega t} \int_{\omega - ir}^{\omega + ir} e^{zt} g(z) \, dz
\]

(1.8)

It follows from (1.7) and (1.8) that

\[
\lim_{m \to \infty} \left( \frac{1}{m} \int_0^m G_{\omega, r} \, dr \right)(t) = \lim_{m \to \infty} \frac{1}{m} \int_0^m G_{\omega, r}(t) \, dr
\]

\[
= e^{-\omega t} PT(t)(\lambda_0 - A)^{-\beta} x_0
\]

for all \( t > 0 \). In the first identity we used the fact that the map \( r \mapsto G_{\omega, r} \) is continuous as a map into \( C_0(\mathbb{R}, Y) \) by the Riemann-Lebesgue lemma. Therefore the integrals with respect to \( r \) can be regarded as Bochner integrals in \( C_0(\mathbb{R}, Y) \) and we may use the continuity of point evaluations.

We also have

\[
\lim_{m \to \infty} \left( \frac{1}{m} \int_0^m G_{\omega, r} \, dr \right) = \lim_{r \to \infty} G_{\omega, r} = G_\omega
\]

in the norm of \( L^q(\mathbb{R}, Y) \). Since norm convergent sequences have pointwise a.e. convergent subsequences, we see that \( G_\omega(t) = e^{-\omega t} PT(t)(\lambda_0 - A)^{-\beta} x_0 \) for almost all \( t > 0 \) and the claim is proved.
It follows that $t \mapsto e^{-\omega t} PT(t)(\lambda_0 - A)^{-\beta} x_0$ defines an element of $L^q(\mathbb{R}_+, Y)$ and

$$\|e^{-\omega \cdot q} PT(\cdot)(\lambda_0 - A)^{-\beta} x_0\|_q \leq \|G_{\omega}\|_q \leq \frac{c_p}{2\pi} \|g_{\omega}\|_p \leq \frac{c_p C}{2\pi}. $$

By the monotone convergence theorem, upon letting $\omega \downarrow 0$ we obtain

$$\|PT(\cdot)(\lambda_0 - A)^{-\beta} x_0\|_q \leq \frac{c_p C}{2\pi}. $$

For $\alpha = 0$, this gives Theorem 0.1.

Let $x_0 \in X$ and $x_0^* \in X^*$ be such that the map $\lambda \mapsto \langle x_0^*, R(\lambda, A)x_0 \rangle$ admits a bounded holomorphic extension to $\{\text{Re} \lambda > 0\}$. Taking $Y = \mathbb{C}$ and $P = x_0^*$, Theorem 1.2 shows that

$$\int_0^\infty |\langle x_0^*, T(t)(\lambda_0 - A)^{-\beta} x_0 \rangle|^q < \infty$$

for all $p \in (1, 2]$, $\beta > \frac{1}{p}$; $\frac{1}{p} + \frac{1}{q} = 1$. This is an individual version of [16, Theorem 5.1], and this observation can be used to show that for $\alpha = 0$ and $p = 2$, the bound $\beta > \alpha + \frac{1}{p} (\alpha = \frac{1}{2})$ in Theorem 1.2 is optimal in the sense that a counterexample exists for all $\beta \in [0, \frac{1}{2})$. Indeed, assume that the theorem holds for $\alpha = 0$, $p = 2$ and some $\beta \geq 0$. Suppose that $T$ is a $C_0$-semigroup on a Banach space $X$ whose resolvent $R(\lambda, A)$ is uniformly bounded in $\{\text{Re} \lambda > 0\}$. Let $\lambda_0 > \max\{0, \omega_0(T)\}$. Then by the observation just made,

$$\int_0^\infty |\langle x_0^*, T(t)(\lambda_0 - A)^{-\beta} x_0 \rangle|^2 < \infty, \quad \forall x \in X, x^* \in X^*.$$  

For each $x \in X$ and $x^* \in X^*$ put

$$f_{x,x^*}(t) := \langle x^*, T(t)(\lambda_0 - A)^{-\beta} x \rangle, \quad t \geq 0.$$  

Then $f_{x,x^*} \in L^2(\mathbb{R}_+)$ and by general considerations involving the closed graph theorem there exists a constant $C > 0$ such that $\|f_{x,x^*}\|_2 \leq C \|x\| \|x^*\|$ for all $x \in X$ and $x^* \in X^*$. By the Plancherel theorem, $s \mapsto \langle x^*, R(is, A)(\lambda_0 - A)^{-\beta} x \rangle \in L^2(\mathbb{R})$. Hence for all $\gamma > \frac{1}{2}$ and $\omega > 0$, by Hölder’s inequality the function

$$g_{\omega,x,x^*}(s) := (\omega + is)^{-\gamma} \langle x^*, R(-is, A)(\lambda_0 - A)^{-\beta} x \rangle$$

belongs to $L^1(\mathbb{R})$. In particular, the Fourier transforms $\mathcal{F} g_{\omega,x,x^*}$ are bounded.

Claim: $\frac{1}{2\pi} \mathcal{F} g_{\omega,x,x^*}(t) = \langle x^*, T(t)(\omega - A)^{-\gamma}(\lambda_0 - A)^{-\beta} x \rangle$ for all $t > 0$. 

Indeed, for $t > 0$ we have, with $A_{\omega} := A - \omega$,

$$\frac{1}{2\pi} \mathcal{F}g_{\omega,x,x^*}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist}(\omega + is)^{-\gamma}\langle x^*, R(-is, A)(\lambda_0 - A)\rangle ds$$

$$= \frac{1}{2\pi} e^{i\omega t} \int_{\text{Re}\lambda = -\omega} e^{\lambda t}(-\lambda)^{-\gamma}\langle x^*, R(\lambda, A_{\omega})(\lambda_0 - A)\rangle d\lambda$$

If $x \in D(A) = D(A_{\omega})$, then by [16, Lemma 3.3] the right most hand equals

$$e^{i\omega t}\langle x^*, T_{\omega}(t)(-A_{\omega})^{-\gamma}(\lambda_0 - A)\rangle^{-\beta}x = \langle x^*, T(t)(\omega - A)^{-\gamma}(\lambda_0 - A)\rangle^{-\beta}x,$$

where $T_{\omega}(t) := e^{-i\omega t}T(t)$. For general $x \in X$, we choose a sequence $x_n \to x$ with $x_n \in D(A)$ for all $n$. Then $f_{x_n,x^*} \to f_{x,x^*}$ in $L^2(\mathbb{R}_+)$ for all $x^* \in X^*$, hence $g_{\omega,x,x^*} \to g_{\omega,x,x^*}$ in $L^1(\mathbb{R})$, and so $\mathcal{F}g_{\omega,x,x^*} \to \mathcal{F}g_{\omega,x,x^*}$ in $C_0(\mathbb{R})$. Therefore, for all $t > 0$,

$$\frac{1}{2\pi} \mathcal{F}g_{\omega,x,x^*}(t) = \lim_{n \to \infty} \frac{1}{2\pi} \mathcal{F}g_{\omega,x,x^*}(t)$$

$$= \lim_{n \to \infty} \langle x^*, T(t)(\omega - A)^{-\gamma}(\lambda_0 - A)^{-\beta}x_n \rangle$$

$$= \langle x^*, T(t)(\omega - A)^{-\gamma}(\lambda_0 - A)^{-\beta}x \rangle.$$
then for all $\beta \geq 0$ with $\beta > \alpha + \frac{1}{p}$ and all $\lambda_0 > \omega_0(T)$ we have
\[
\lim_{t \to \infty} \|T(t)(\lambda_0 - A)^{-\beta}x_0\| = 0.
\]

Proof: By Theorem 1.2 applied to the case $Y = X$ and $P = I$ we find that the function $f(t) := T(t)(\lambda_0 - A)^{-\beta}x_0$ defines an element of $L^q(\mathbb{R}_+, X)$, $\frac{1}{p} + \frac{1}{q} = 1$. Hence a standard argument (cf. the proof of [17, Theorem 4.4.1]) shows that $\lim_{t \to \infty} \|f(t)\| = 0$. \///

Recalling that a $B$–convex Banach space $X$ has non-trivial Fourier type, we see from Corollary 1.3 that
\[
\lim_{t \to \infty} \|T(t)R(\lambda, A)x_0\| = 0
\]
whenever $T$ is a $C_0$–semigroup on a $B$–convex space $X$ and $x_0 \in X$ is such that the local resolvent $R(\lambda, A)x_0$ admits a bounded holomorphic extension to the open right half-plane. This improves the result of [9] mentioned in the introduction.

We next discuss the analogue of Corollary 1.3 for general operators $P$. Although the proof of Corollary 1.3 breaks down, for slightly larger values of $\beta$ we can prove:

**Theorem 1.4.** Let $P$ be a bounded operator from a Banach space $X$ into a $B$–convex Banach space $Y$. Let $A$ be the generator of a $C_0$–semigroup $T$ on $X$ and let $x_0 \in X$ be such that the map $\lambda \mapsto PR(\lambda, A)x_0$ extends to a holomorphic function $F(\lambda)$ in the open right half-plane. If there exist $\omega_0 > \max\{0, \omega_0(T)\}$, $M > 0$ and $\alpha \in [-1, \infty)$ such that
\[
\|F(\lambda)\| \leq M(1 + |\lambda|)^\alpha, \quad 0 < \text{Re} \lambda < \omega_0,
\]
then for all $\beta > \alpha + 1$ and $\lambda_0 > \max\{0, \omega_0(T)\}$ we have
\[
\lim_{t \to \infty} \|PT(t)(\lambda_0 - A)^{-\beta}x_0\| = 0.
\]

Proof: Without loss of generality we may assume that $\omega_0(T) \geq 0$. Fix $\lambda_0 > \omega_0(T)$. Let $p \in (1, 2]$ be the Fourier type of $Y$. Then $Y$ has also Fourier type $p'$ for all $p' \in (1, p]$. Hence, since $\beta > 0$ by assumption, upon replacing $p$ by a smaller value we may assume that $\beta > \frac{1}{q}, \frac{1}{p} + \frac{1}{q} = 1$. This enables us to choose $\delta \geq 0$ such that $\delta > \alpha + \frac{1}{p}$ in such a way that $\frac{1}{q} < \gamma := \beta - \delta < 1$. Consider the functions
\[
f(t) := PT(t)(\lambda_0 - A)^{-\delta}x_0, \quad g(t) := PT(t)(\lambda_0 - A)^{-\beta}x_0; \quad t \geq 0.
\]
By Theorem 1.2, $f \in L^q(\mathbb{R}_+, Y)$. For $t \geq 0$ we have

$$
g(t) = PT(t)(\lambda_0 - A)^{-\delta} x_0 = PT(t)(\lambda_0 - A)^{-\delta} \left( \frac{\sin \pi \gamma}{\pi} \int_0^\infty s^{-\gamma} R(\lambda_0 + s, A) x_0 ds \right)
$$

$$
= \frac{\sin \pi \gamma}{\pi} P(\lambda_0 - A)^{-\delta} \int_0^\infty s^{-\gamma} \int_0^\infty e^{-(\lambda_0 + r)s} T(t + r) x_0 dr ds
$$

(1.9)

Now,

$$
\left\| \int_0^\infty e^{-(\lambda_0 + r)s} f(t + r) x_0 dr \right\| \leq \left( \int_0^\infty e^{-(\lambda_0 + r)p} dr \right)^{\frac{1}{p}} \left( \int_0^\infty \| f(t + r) \|^q dr \right)^{\frac{1}{q}}
$$

$$
= \frac{1}{(p(\lambda_0 + s))^{\frac{1}{p}}} \left( \int_t^\infty \| f(r) \|^q dr \right)^{\frac{1}{q}}.
$$

Combining this estimate with (1.9) yields

$$
\|g(t)\| \leq \frac{\sin \pi \gamma}{\pi p^{\frac{1}{q}}} \int_0^\infty s^{-\gamma}(\lambda_0 + s)^{-\frac{1}{p}} ds \cdot \left( \int_t^\infty \| f(r) \|^q dr \right)^{\frac{1}{q}}.
$$

Since $\frac{1}{q} < \gamma < 1$, the first integral in the above expression is absolutely convergent, and the second tends to 0 as $t \to \infty$. This proves that $\lim_{t \to \infty} \|g(t)\| = 0$. ////

Theorem 0.3 is a special case of Theorem 1.4 by taking $\alpha = 0$, $Y = \mathbb{C}$, and $P = x^*$. Of course, Theorem 0.3 can be proved without reference to $B$-convexity: Take $Y = X$ and $P = x^*$ in the proofs of Theorems 1.2 and 1.5 and use the Hausdorff-Young theorem instead of the Fourier type. A similar remark applies to Corollary 2.3 below.

For $\alpha = 0$, Theorem 1.4 fails for every $0 \leq \beta < 1$ (the case $\beta = 1$ remains open). Indeed, consider the case that the resolvent $R(\lambda, A)$ itself is uniformly bounded in $\{\text{Re} \lambda > 0\}$. Then the assumptions of Theorem 1.4 are satisfied for $\alpha = 0$, all $x_0 \in X$, and all functionals $P = x^* \in X^*$. Hence if the theorem holds for some $\beta \geq 0$, then from the uniform boundedness principle we conclude

$$
\sup_{t \geq 0} \| T(t)(\lambda_0 - A)^{-\beta} \| < \infty.
$$

For $0 \leq \beta < 1$, this contradicts the example in [22] cited in the discussion after Theorem 1.2.

We next turn to a version of Theorem 1.4 which holds for $\beta > \alpha + \frac{1}{p}$ rather than $\beta > \alpha + 1$. The price for this is the a priori assumption that $PT(\cdot)x_0$ is bounded.
Theorem 1.5. Let $P$ be a bounded linear operator from a Banach space $X$ into a Banach space $Y$ with Fourier type $p \in (1,2]$. Let $A$ be the generator of a $C_0$-semigroup $T$ on $X$ and let $x_0 \in X$ be such that the orbit $t \mapsto PT(t)x_0$ is bounded and $\lambda \mapsto PR(\lambda, A)x_0$ admits a holomorphic extension $F(\lambda)$ to the open right half-plane. If there exist $\omega_0 > \max\{0, \omega_0(T)\}$, $M > 0$ and $\alpha \in [-1,\infty)$ such that

$$\|F(\lambda)\| \leq M(1 + |\lambda|)^\alpha, \quad 0 < \text{Re} \lambda < \omega_0,$$

then for all $\lambda_0 > \omega_0(T)$ and $\beta \geq 1$ with $\beta > \alpha + \frac{1}{p}$ we have

$$\lim_{t \to \infty} \|PT(t)(\lambda_0 - A)^{-\beta}x_0\| = 0.$$

Proof: Without loss of generality we may assume that $\omega_0(T) \geq 0$. Fix $\lambda_0 > \omega_0(T)$ and $\beta \geq 1$ with $\beta > \alpha + \frac{1}{p}$. For each $\delta \geq 0$ consider the function

$$f_\delta(t) := PT(t)(\lambda_0 - A)^{-\delta}x_0, \quad t \geq 0.$$

We have to show that $\lim_{t \to \infty} \|f_\delta(t)\| = 0$. Theorem 1.2 shows that $f_\delta \in L^q(\mathbb{R}, Y)$, $\frac{1}{p} + \frac{1}{q} = 1$.

Let $\delta = n + \gamma$ with $n \in \mathbb{N}$ and $\gamma \in [0,1)$. If $\gamma \in (0,1)$, then

$$\|PT(\tau)(\lambda_0 - A)^{-\gamma}x_0\| = \frac{\sin \pi \gamma}{\gamma} \left\| \int_0^\infty r^{-\gamma} PT(\tau) R(\lambda_0 + r, A)x_0 \, dr \right\|$$

$$= \frac{\sin \pi \gamma}{\gamma} \left\| \int_0^\infty r^{-\gamma} \int_0^\infty e^{-(\lambda_0 + r)s} PT(\tau + s)x_0 \, ds \, dr \right\|$$

$$\leq \frac{\sin \pi \gamma}{\gamma} \int_0^\infty C r^{-\gamma} (\lambda_0 + r)^{-1} \, dr,$$

where $C := \sup_{t \geq 0} \|PT(t)x_0\|$. If $\gamma = 0$, then $\|PT(\tau)x_0\| \leq C$. In either case, we see that $C_\gamma := \sup_{r \geq 0} \|PT(\tau)(\lambda_0 - A)^{-\gamma}x_0\| < \infty$. Using this, we obtain

$$\|f_\delta(t)\| = \left\| \int_0^\infty \cdots \int_0^\infty e^{-\lambda_0(s_1 + \ldots + s_n)} PT(t + s_1 + \ldots + s_n)(\lambda_0 - A)^{-\gamma}x_0 \, ds_n \ldots ds_1 \right\|$$

$$\leq C_\gamma \lambda_0^{-n}$$

for all $t \geq 0$, so $f_\delta$ is bounded. In particular, such an estimate holds for $f_\beta$. Also, $f_\beta$ is differentiable and

$$f'_\beta(t) = PT(t)A(\lambda_0 - A)^{-\beta}x_0 = -f_{\beta-1}(t) + \lambda_0 f_\beta(t).$$

Therefore, also $f_\beta(\cdot)$ is bounded (here we use that $\beta \geq 1$) and hence the bounded function $f_\beta(\cdot)$ is uniformly continuous. Then also $\|f_\beta(\cdot)\|^q = \|PT(\cdot)(\lambda_0 - A)^{-\beta}x_0\|^q$ is bounded and uniformly continuous, and it is an immediate consequence of Theorem 1.2 that $\|f_\beta(t)\| \to 0$ as $t \to \infty$. ///
Assuming boundedness and uniform continuity of $PT(\cdot)x_0$, we obtain a stronger result. Let us say that a function $F$ is polynomially bounded in the strip $\{0 < \text{Re} \, \lambda < \omega_0\}$ if there exist $M > 0$ and $n \in \mathbb{N}$ such that

$$\|F(\lambda)\| \leq M(1 + |\lambda|)^n, \quad 0 < \text{Re} \, \lambda < \omega_0. \quad (1.10)$$

**Corollary 1.6.** Let $P$ be a bounded linear operator from $X$ into a $B$-convex space $Y$. Let $A$ be the generator of a $C_0$-semigroup $T$ on $X$ and let $x_0 \in X$ be such that the orbit $t \mapsto PT(t)x_0$ is bounded and uniformly continuous. If the map $\lambda \mapsto PR(\lambda, A)x_0$ extends to a holomorphic function in the open right half-plane which is polynomially bounded in $\{0 < \text{Re} \, \lambda < \omega_0\}$ for some $\omega_0 > \max\{0, \omega_0(T)\}$, then $\lim_{t \to \infty} \|PT(t)x_0\| = 0$.

**Proof:** Fix $\lambda > \omega_0(T)$. Let $S$ denote the left translation semigroup on the space $Z := BUC(\mathbb{R}_+, Y)$ defined by $(S(t)f)(s) = f(t+s); s, t \geq 0$. The function $f(t) := PT(t)x_0$ defines an element of $Z$. From the identity

$$PT(t)R(\lambda, A)^{n+1}x_0 = \int_0^\infty \cdots \int_0^\infty e^{-\lambda(s_1+\cdots+s_{n+1})}PT(s_1+\cdots+s_{n+1}+t)x_0 \, ds_{n+1} \cdots ds_1$$

it is easy to see that also $f_\lambda(t) := PT(t)R(\lambda, A)^{n+1}x_0$ defines an element of $Z$; here $n \in \mathbb{N}$ is chosen such that (1.10) holds.

By Theorem 1.5,

$$\lim_{t \to \infty} \|S(t)f_\lambda\| = \lim_{t \to \infty} \left( \sup_{s \geq 0} \|PT(t+s)R(\lambda, A)^{n+1}x_0\| \right) = 0.$$ 

Therefore, $f_\lambda \in Z_0 := \{f \in Z : \lim_{t \to \infty} \|S(t)f\|_Z = 0\}$. For $\text{Re} \, \lambda > \omega_0(T)$ and $s \geq 0$ we have, denoting by $B$ the generator of $S$,

$$(R(\lambda, B)^{n+1}f)(s) = \int_0^\infty \cdots \int_0^\infty e^{-\lambda(t_1+\cdots+t_{n+1})}(S(t_1+\cdots+t_{n+1})f)(s) \, dt_{n+1} \cdots dt_1$$

$$= \int_0^\infty \int_0^\infty e^{-\lambda(t_1+\cdots+t_{n+1})}PT(t_1+\cdots+t_{n+1}+s)x_0 \, dt_{n+1} \cdots dt_1$$

$$= PT(s)R(\lambda, A)^{n+1}x_0 = f_\lambda(s).$$

Hence $f = \lim_{\lambda \to \infty} \lambda^{n+1}R(\lambda, B)^{n+1}f = \lim_{\lambda \to \infty} \lambda^{n+1}f_\lambda \in Z_0$ by the closedness of $Z_0$.

Hence $\lim_{t \to \infty} \|S(t)f\| = 0$, and thus $\lim_{t \to \infty} \|PT(t)x_0\| = \lim_{t \to \infty} \|(S(t)f)(0)\| = 0$.  

The technique of this proof goes back to Kantorovitz [10]; see [2] for another application.

The following example shows that our results break down if no restrictions on the Banach space $X$ are imposed.
**Example 1.7.** Let $X = C_0(\mathbb{R})$ and consider the left translation group $S$ on $X$. Let $B$ be its generator. Let $f \in X$ be any non-zero function with support in $[0, 1]$. Then for all $\Re \lambda > 0$ and $s \in \mathbb{R}$ we have

$$|(R(\lambda, B)f)(s)| = \left| \int_0^\infty e^{-\lambda t} f(s + t) \, dt \right| \leq \|f\|_{\infty}.$$

Consequently,

$$\sup_{\Re \lambda > 0} \|R(\lambda, B)f\|_{\infty} \leq \|f\|_{\infty},$$

but since $S$ is isometric and $(\lambda_0 - B)^{-\beta}$ is injective we see that

$$\lim_{t \to \infty} \|S(t)(\lambda_0 - B)^{-\beta} f\|_{\infty} = \|(\lambda_0 - B)^{-\beta} f\|_{\infty} \neq 0; \quad \forall \beta \geq 0, \lambda_0 > 0.$$

As an application of Corollary 1.6 we shall derive a Tauberian theorem for the Laplace transform of functions in $L^1(\mathbb{R}_+, Y)$, where $Y$ is a $B-$convex Banach space. This serves merely as an illustration of what can be done with the above theory; by considering bounded, uniformly continuous orbits much of the sharpness of the preceding results is lost and it may well be that more direct methods will lead to a sharper Tauberian theorem (cf. the remarks at the end of the paper).

**Lemma 1.8.** Let $Y$ be a $B-$convex Banach space and assume that the Laplace transform $\hat{g}$ of a function $g \in BUC(\mathbb{R}_+, Y)$ is polynomially bounded in some strip $\{0 < \Re \lambda < \omega_0\}$. Then $\lim_{t \to \infty} \|g(t)\| = 0$.

**Proof:** Consider the left translation semigroup $S$ in $BUC(\mathbb{R}_+, Y)$ with generator $B$. Let $P$ be the bounded operator from $BUC(\mathbb{R}_+, Y)$ into $Y$ defined by $Ph = h(0)$. Then $PS(t)g = g(t) \otimes 1$ and $PB(\lambda,B)g = \hat{g}(\lambda) \otimes 1$ for all $t \geq 0$ and $\Re \lambda > 0$. Since $Y$ is $B-$convex, we can apply Corollary 1.6 to $S$ and deduce that $\lim_{t \to \infty} \|g(t)\| = \lim_{t \to \infty} \|PS(t)g\| = 0$. \\

**Theorem 1.9.** Let $Y$ be a $B-$convex Banach space and let $f \in L^\infty(\mathbb{R}_+, Y)$. If the Laplace transform $\hat{f}$ is polynomially bounded in some strip $\{0 < \Re \lambda < \omega_0\}$ and can be holomorphically extended to a neighbourhood of 0, then

$$\lim_{t \to \infty} \left\| \int_0^t f(s) \, ds - \hat{f}(0) \right\| = 0.$$

**Proof:** The proof is inspired by [2, Theorem 4.3].

Upon replacing $f(t)$ by $f(t) - e^{-t}\hat{f}(0)$ we may assume that $\hat{f}(0) = 0$. By a special case of Ingham’s Tauberian theorem the function $g(t) := \int_0^t f(s) \, ds$ is bounded (see [11] for an elegant and elementary proof). Moreover, $g$ is uniformly continuous and in
view of \( \hat{g}(0) = 0 \), 0 is a removable singularity of \( \hat{g}(\lambda) = \lambda^{-1} \hat{f}(\lambda) \). It follows that \( \hat{g} \) is polynomially bounded in \( \{0 < \text{Re} \lambda < \omega_0 \} \). Therefore by Lemma 1.8,

\[
\lim_{t \to \infty} \left\| \int_0^t f(s) \, ds \right\| = \lim_{t \to \infty} \| g(t) \| = 0.
\]

2. Stability and the analytic Radon-Nikodym property

In this section we will prove some analogues of the previous results for the case \( p = 1 \). As it turns out, this is possible if one assumes \( Y \) has the analytic Radon-Nikodym property.

We start by recalling some facts concerning vector-valued Hardy spaces over the disc \( D = \{ z \in \mathbb{C} : |z| < 1 \} \).

For \( p \in [1, \infty] \) we let \( H^p(D, Y) \) denote the set of all holomorphic functions \( f : D \to Y \) for which

\[
\| f \|_p := \sup_{0 < r < 1} \left( \int_0^{2\pi} \| f(re^{i\theta}) \|^p \, d\theta \right)^{\frac{1}{p}} < \infty.
\]

In case \( p = \infty \) we interpret the above integral in terms of the supremum norm in the obvious way. It is not difficult to see that \( H^p(D, Y) \) is a Banach space with respect to the norm \( \| \cdot \|_p \). We let \( H^p_0(D, Y) \) denote the closed subspace of \( H^p(D, Y) \) consisting of all functions \( f \) for which the radial limits \( \tilde{f}(e^{i\theta}) := \lim_{r \to 1} f(re^{i\theta}) \) exist for almost all \( \theta \). By Fatou’s lemma,

\[
\int_0^{2\pi} \| \tilde{f}(e^{i\theta}) \|^p \, d\theta \leq \liminf_{r \to 1} \int_0^{2\pi} \| f(re^{i\theta}) \|^p \, d\theta,
\]

which shows that the boundary function \( \tilde{f} \), if it exists a.e., belongs to \( L^p(\Gamma) \), where \( \Gamma = \{ z \in \mathbb{C} : |z| = 1 \} \). In this case, \( f \) can be recovered from \( \tilde{f} \) by the Poisson integral

\[
f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(e^{i\eta}) \frac{1 - r^2}{1 - 2r \cos(\theta - \eta) + r^2} \, d\eta.
\]

Defining \( f_r(e^{i\theta}) := f(re^{i\theta}) \), as in the scalar case it follows from this representation that

\[
\lim_{r \to 1} \| \tilde{f} - f_r \|_{L^p(\Gamma)} = 0.
\]
A Banach space $Y$ is said to have the analytic Radon-Nikodym property if $H^0_0(D,Y) = H^p(D,Y)$. Equivalently, $Y$ has the analytic Radon-Nikodym property if for all $f \in H^p(D,Y)$ the radial limits $\tilde{f}(e^{i\theta}) := \lim_{r \uparrow 1} f(re^{i\theta})$ exist for almost all $\theta$, and in this case we actually have $f_r \to \tilde{f}$ in the $L^p$-norm.

The role of the exponent $p$ needs some clarification: it can be shown that if $H^0_0(D,Y) = H^p(D,Y)$ holds for some $p \in [1, \infty]$, then it holds for all $p \in [1, \infty]$.

The following facts are well-known:

(i) If $Y$ has the Radon-Nikodym property, then $Y$ has the analytic Radon-Nikodym property;
(ii) If $Y$ has the analytic Radon-Nikodym property, then $Y$ contains no closed subspace isomorphic to $c_0$;
(iii) A Banach lattice $Y$ has the analytic Radon-Nikodym property if and only if $Y$ contains no closed subspace isomorphic to $c_0$.

It follows from (i) that every reflexive Banach space and every separable dual Banach space has the analytic Radon-Nikodym property. By (iii), the spaces $L^1(\mu)$ have the analytic Radon-Nikodym property. The proofs can be found in [5,6].

By mapping a rectangle conformally onto the unit disc it is not difficult to prove the following result; cf. [7].

**Proposition 2.1.** Let $\Delta$ and $\Delta_r$, $0 < r < 1$, be the rectangles in $\mathbb{C}$ spanned by the points $\pm a \pm ib$ and $\pm ra \pm irb$, respectively. Let $f$ be a holomorphic $Y$-valued function in the interior of $\Delta$. Assume that $Y$ has the analytic Radon-Nikodym property and that
\[
\sup_{0 < r < 1} \int_{\Delta_r} \|f(z)\| |dz| < \infty.
\]
Then, the strong limits $\lim_{r \uparrow 1} f(rz)$ exist for almost all $z \in \Delta$ and define a function $\tilde{f} \in L^1(\Delta)$. Moreover,
\[
\lim_{r \uparrow 1} \int_\Delta \|\tilde{f}(z) - f(rz)\| |dz| = 0.
\]

///

**Theorem 2.2.** Let $P$ be a bounded operator from a Banach space $X$ into a Banach space $Y$ with the analytic Radon-Nikodym property. Let $A$ be the generator of a $C_0$-semigroup $\mathbf{T}$ on $X$. Assume that for some $x_0 \in X$, the map $\lambda \mapsto PR(\lambda, A)x_0$ admits a holomorphic extension $F(\lambda)$ to the open right half-plane. If there exist $\omega_0 > \max\{0, \omega_0(\mathbf{T})\}$, $M > 0$ and $\alpha \in [-1, \infty)$ such that
\[
\|F(\lambda)\| \leq M(1 + |\lambda|)^\alpha, \quad 0 < \text{Re} \lambda < \omega_0,
\]
then for all \( \lambda_0 > \max\{0, \omega_0(T)\} \) and \( \beta > \alpha + 1 \) we have
\[
\lim_{t \to \infty} \| PT(t)(\lambda_0 - A)^{-\beta}x_0 \| = 0.
\]

Proof: Without loss of generality we may assume that \( \omega_0(T) \geq 0 \). Fix \( \lambda_0 > \omega_0(T) \). By taking a smaller value of \( \omega_0 \) we may assume that \( \omega_0(T) < \omega_0 < \lambda_0 \).

Fix \( \gamma \in (\alpha + 1, \beta) \) and let \( \delta := \beta - \gamma \).

Let \( g(\lambda) \) denote the holomorphic extension in the open right half-plane of the function \( \lambda \mapsto PR(\lambda, A)(\lambda_0 - A)^{-\gamma}x_0 \). Fix \( \omega_1 \in (\omega_0(T), \omega_0) \). On the strip \( \{ 0 < \Re \lambda < \omega_1 \} \) we define \( h(\lambda) := (\omega_0 - \lambda)^{-\delta} g(\lambda) \). By Lemma 1.1, for each \( \zeta \in \mathbb{C} \) with \( 0 < \Re \zeta < \omega_1 \) the function
\[
s \mapsto h_\zeta(s) := h(\zeta - is) = (\omega_0 - \zeta + is)^{-\delta} g(\zeta - is)
\]
belongs to \( L^1(\mathbb{R}, Y) \), and the map \( \zeta \mapsto h_\zeta \) is a bounded \( L^1(\mathbb{R}, Y) \)-valued holomorphic function on \( \{ 0 < \Re \zeta < \omega_1 \} \).

Arguing as in the proof of the Claim following Theorem 1.2 we see that for \( \omega \in (\omega_0(T), \omega_1) \) the Fourier transform of \( h_\omega \) is given by
\[
\frac{1}{2\pi} \mathcal{F} h_\omega(t) = e^{-\omega t} PT(t)(\omega_0 - A)^{-\delta} (\lambda_0 - A)^{-\gamma}x_0.
\]

Hence by uniqueness of analytic continuation,
\[
\frac{1}{2\pi} \mathcal{F} h_\zeta(t) = e^{-\zeta t} PT(t)(\omega_0 - A)^{-\delta} (\lambda_0 - A)^{-\gamma}x_0, \quad 0 < \Re \zeta < \omega_1,
\]
and we conclude that (2.1) holds for all \( \omega \in (0, \omega_1) \).

Since \( Y \) has the analytic Radon-Nikodym property, we may apply Proposition 2.1 and conclude that the boundary function \( \hat{h} \) of \( h \) exists a.e. on \( i\mathbb{R} \), defines an element in \( L^1_{loc}(i\mathbb{R}, Y) \), and that
\[
\lim_{\omega \downarrow 0} \int_{-r}^{r} \| \hat{h}(is) - h(\omega + is) \| \, ds = 0
\]
for all \( r > 0 \). But then (1.3) and the definition of \( h \) easily implies that we actually have \( \hat{h} \in L^1(i\mathbb{R}, Y) \) and
\[
\lim_{\omega \downarrow 0} \int_{-\infty}^{\infty} \| \hat{h}(is) - h(\omega + is) \| \, ds = 0.
\]

Hence by passing to the limit \( \omega \downarrow 0 \) in (2.1), we obtain
\[
PT(\cdot)(\omega_0 - A)^{-\delta} (\lambda_0 - A)^{-\gamma}x_0 = \lim_{\omega \downarrow 0} \frac{1}{2\pi} \lim_{\omega \downarrow 0} PT(\cdot)(\omega_0 - A)^{-\delta} (\lambda_0 - A)^{-\beta}x_0
\]
\[
= \frac{1}{2\pi} \lim_{\omega \downarrow 0} \mathcal{F} h(\omega - i(\cdot))(t) = \frac{1}{2\pi} \mathcal{F} \hat{h}(-i(\cdot))(t).
\]
Therefore, \( PT(\cdot)(\omega_0 - A)^{-\delta} (\lambda_0 - A)^{-\gamma}x_0 \in C_{0}(\mathbb{R}_+, Y) \) by the Riemann-Lebesgue lemma. Recalling that \( \delta + \gamma = \beta \), by standard arguments involving fractional powers this will give the desired result. ///
Theorem 0.2 is a special case of this.

Taking $Y = \mathbb{C}$ and $P := x_0^* \in X^*$, we obtain the following result, which contains Theorem 0.3 as a special case.

**Corollary 2.3.** Let $A$ be the generator of a $C_0$-semigroup $T$ on a Banach space $X$. Assume that for some $x_0 \in X$ and $x_0^* \in X^*$, the map $\lambda \mapsto \langle x_0^*, R(\lambda, A)x_0 \rangle$ admits a holomorphic extension $F(\lambda)$ to the open right half-plane. If there exist $\omega_0 > \max\{0, \omega_0(T)\}$, $M > 0$ and $\alpha \in [-1, \infty)$ such that

$$|F(\lambda)| \leq M(1 + |\lambda|)^\alpha, \quad 0 < \Re \lambda < \omega_0,$$

then for all $\lambda_0 > \max\{0, \omega_0(T)\}$ and $\beta \geq 0$ with $\beta > \alpha + 1$ we have

$$\lim_{t \to \infty} (x_0^*, T(t)(\lambda_0 - A)^{-\beta}x_0) = 0.$$

The case $\alpha = 0$ of Theorem 2.2 can be used to show that Corollary 1.6, and therefore also Theorem 1.9, remains valid if $B$–convexity is replaced by the analytic Radon-Nikodym property. It is possible, however, to modify the proof of [11] to prove in a more direct way the stronger result: if $Y$ has the analytic Radon-Nikodym property and $f \in L^\infty(\mathbb{R}_+, Y)$ is such that for all $r > 0$ we have

$$\limsup_{\omega \to 0} \int_{-r}^r \left\| \frac{\hat{f}(\omega + is) - \hat{f}(0)}{\omega + is} \right\| ds < \infty,$$

then $\lim_{t \to \infty} \left\| \int_0^t f(s) ds - \hat{f}(0) \right\| = 0$. This was shown by Chill [7] and suggests that it may be possible to prove a similar result assuming $B$–convexity. It is important in this context to point out that $B$–convexity and the analytic Radon-Nikodym property are unrelated concepts in the sense that none implies the other. In fact, $L^1[0, 1]$ has the analytic Radon-Nikodym property (by observation (iii) at the beginning of this section) but no non-trivial type, so it is not $B$–convex. The following example shows that there exist $B$–convex spaces without the analytic Radon-Nikodym property:

**Example 2.4.** By the function space analogue of a result in [20] (the details are given in [24]), the operator of integration $I : L^1[0, 1] \to C[0, 1]$, $I(f)(t) := \int_0^t f(s) ds$, factors through a space with type 2. Denoting $f_0(t) := t$ and defining $T : C[0, 1] \to C[0, 1]$ by $T(f) := f - f(1)f_0$, also $J := T \circ I$ factors through a space with type
2. Identifying $[0, 1)$ with the unit circle $\Gamma$ in the complex plane and letting $e_n(\theta) := \exp(2\pi in\theta)$, $\theta \in \Gamma$, $n \in \mathbb{Z}$, we can represent $J$ as an operator from $L^1(\Gamma)$ into $C(\Gamma)$ by
\[
J(e_n) = e_n/(2\pi i n), \quad n \in \mathbb{Z} \setminus \{0\}, \quad J(e_0) = 0.
\]
Recalling that type passes to quotients, it follows that the quotient operator $J_0 : L^1(\Gamma)/H^1_0 \to C(\Gamma)/A_0$ induced by $J$ factors through a space with type 2; here $H^1_0$ and $A_0$ denote the closed linear span in $L^1(\Gamma)$ and $C(\Gamma)$, respectively, of $\{\theta \mapsto \exp(2\pi i n\theta) : n = -1, -2, \ldots\}$. On the other hand, by a result of Pisier [8, Proposition V.5], $J_0$ cannot be factored through a space with the analytic Radon-Nikodym property.

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Note added in proof - Recently, V. Wrobel [23] has shown that the bound $\beta > \frac{1}{p}$ in Theorem 0.1 is the best possible, in the sense that a counterexample can be constructed for every $\beta \in [0, \frac{1}{p})$. Whether or not the theorem holds for $\beta = \frac{1}{p}$ remains an open problem. In the same paper, an extension Theorem 0.1 into a different direction is obtained.

References


