STOCHASTIC INTEGRATION IN UMD SPACES

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We report on a joint work with Mark Veraar and Lutz Weis [6].

Building upon previous work by Rosiński and Suchanecki [8] and Brzeźniak and the author [1], a systematic theory of stochastic integration for Banach space-valued functions with respect to Brownian motions has been constructed in [7] using a recent idea of Kalton and Weis to study vector-valued functions through certain operator-theoretic properties of the associated integral operators [4]. In the work presented here, the results of [7] are extended to a theory of stochastic integration for stochastic processes taking values in a UMD space.

Let \( (\gamma_n) \) be a sequence of independent standard Gaussian random variables on some probability space \((\Omega, \mathbb{P})\). A bounded operator \( T : H \to E \) acting from a separable real Hilbert space \( H \) with orthonormal basis \( (h_n) \) into a real Banach space \( E \) is said to be \( \gamma \)-radonifying if the Gaussian sum \( \sum_n \gamma_n Th_n \) converges in \( L^2(\Omega; E) \). This definition is independent of the choice of \( (\gamma_n) \) and \( (h_n) \), and the vector space \( \gamma(H, E) \) of all \( \gamma \)-radonifying operators from \( H \) to \( E \) is a Banach space with respect to the norm \( \|T\|_{\gamma(H, E)} \) defined by

\[
\|T\|_{\gamma(H, E)}^2 := \mathbb{E} \left\| \sum_n \gamma_n Th_n \right\|^2.
\]

Let \( W = (W(t))_{t \geq 0} \) be a Brownian motion on \((\Omega, \mathbb{P})\). The main result of [7] can be formulated as follows.

**Theorem 1 ([7]).** For a function \( \psi : [0, T] \to E \) such that \( \langle \psi, x^* \rangle \in L^2(0, T) \) for all \( x^* \in E^* \), the following assertions are equivalent:

1. For every measurable set \( A \subseteq [0, T] \) there exists an \( E \)-valued random variable \( \eta_A \) such that for all \( x^* \in E^* \) we have

\[
\langle \eta_A, x^* \rangle = \int_A \langle \phi(t), x^* \rangle dW(t) \text{ almost surely};
\]

2. There exists an operator \( S_\psi \in \gamma(L^2(0, T), E) \) such that for all \( f \in L^2(0, T) \) and \( x^* \in E^* \) we have

\[
\langle S_\psi f, x^* \rangle = \int_0^T f(t) \langle \psi(t), x^* \rangle dt.
\]

Writing \( \eta_A = \int_A \psi(t) dW(t) \), for all \( 1 \leq p < \infty \) we have \( \mathbb{E} \left\| \int_0^T \psi(t) dW(t) \right\|^p \asymp_p \|S_\psi\|_{\gamma(L^2(0, T), E)}^p \), with equality for \( p = 2 \).

If the equivalent conditions of the theorem are satisfied, then \( \psi \) is said to be stochastically integrable with respect to \( W \).

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Denote by $\mathcal{F}^W_t = (\mathcal{F}^W_t)_{t\geq 0}$ the augmented filtration generated by $W$. A stochastic process $\phi: [0,T] \times \Omega \to E$ is said to be $\mathcal{F}^W$-weakly progressive if for all $x^* \in E^*$ the real-valued process $\langle \phi, x^* \rangle$ is progressively measurable with respect to $\mathcal{F}^W$. Such a process is said to be elementary progressive if it is of the form $\phi = \sum_{n=1}^{\infty} 1_{(t_n, t_{n+1})} \otimes \xi_n$, where $\xi_n$ is an $\mathcal{F}^W_{t_n}$-measurable simple random variable with values in $E$. Assuming that $E$ is a UMD space, Garling [3] proved the following two-sided decoupling inequality for elementary progressive processes, valid for $1 < p < \infty$:

$$
\mathbb{E}_{\Omega} \left\| \int_0^T \phi(t) \, dW(t) \right\|_p^p \approx_{p,E} \mathbb{E}_{\Omega} \left( \mathbb{E}_{\Omega} \left\| \int_0^T \phi(t) \, d\bar{W}(t) \right\|_p^p \right)^{p/2} = \mathbb{E}_\Omega \|S_\phi\|_{\gamma(L^2(0,T),E)}^p,
$$

where $S_\phi: \Omega \to \gamma(L^2(0,T),E)$ satisfies $\langle S_\phi(\omega), f^* \rangle = \int_0^T \langle f(t, \omega), x^* \rangle \, dt$ for all $f \in L^p(0,T)$ and $x^* \in E^*$ almost surely. As a consequence, the mapping $S_\phi \mapsto \int_0^T \phi(t) \, dW(t)$ extends to an isomorphism from the closure in $L^p(\Omega; \gamma(L^2(0,T),E))$ of the elementary progressive processes onto a certain closed subspace of $L^p(\Omega; E)$. Using a version of the Pettis measurability theorem for $\mathcal{F}^W$-measurable processes in combination with Itô’s martingale representation theorem and approximation arguments, the range of this isomorphism can be identified as the subspace of all mean zero $\mathcal{F}^W_T$-measurable elements of $L^p(\Omega; E)$. The result is an extension of Itô’s martingale representation theorem to UMD-valued processes, which is the main ingredient in the proof of the following theorem:

**Theorem 2.** Let $E$ be a UMD space and let $p \in (1, \infty)$. For a weakly progressive process $\phi: [0,T] \times \Omega \to E$ such that $\langle \phi, x^* \rangle \in L^p(\Omega; L^2(0,T))$ for all $x^* \in E^*$, the following assertions are equivalent:

1. For every measurable set $A \subseteq [0,T]$ there exists a random variable $\eta_A \in L^p(\Omega; E)$ such that for all $x^* \in E^*$ we have

$$
\langle \eta_A, x^* \rangle = \int_A \langle \phi(t), x^* \rangle \, dW(t) \quad \text{in } L^p(\Omega);
$$

2. There exists a random variable $S_\phi \in L^p(\Omega; \gamma(L^2(0,T),E))$ such that for all $f \in L^2(0,T)$ and $x^* \in E^*$ we have

$$
\langle S_\phi(\omega), f^* \rangle = \int_0^T f(t) \langle \phi(t, \omega), x^* \rangle \, dt \quad \text{for almost all } \omega \in \Omega.
$$

Writing $\eta_A = \int_A \phi(t) \, dW(t)$ we have $\mathbb{E} \left\| \int_0^T \phi(t) \, dW(t) \right\|_p^p \approx_{p,E} \mathbb{E} \|S_\phi\|_{\gamma(L^2(0,T),E)}^p$.

If the equivalent conditions of the theorem are satisfied, then $\phi$ is said to be $L^p$-stochastically integrable with respect to $W$. Note that the scalar stochastic integral on the right hand side in (1) is well defined in $L^p(\Omega)$ by the Burkholder-Davis-Gundy inequalities. When combined with our generalized Itô representation theorem, the equivalence of norms in the last line of the theorem leads to Burkholder-Davis-Gundy inequalities for UMD-valued $\mathcal{F}^W$-martingales.
Theorem 2 can be applied to show that every continuous $L^p$-martingale $(M_t)_{t \geq 0}$ with respect to the filtration $\mathcal{F}^W$, with values in a UMD space $E$ and satisfying $M_0 = 0$, is $L^p$-stochastically integrable with respect to $W$ on every interval $[0, T]$ and satisfies
$$\mathbb{E}\left[\left\| \int_0^T M_t \, dW(t) \right\|^p \right] \lesssim_{p,E} T^{\frac{2}{p}} \mathbb{E}\|M_T\|^p.$$
In particular this applies to the continuous $L^p$-martingale $M_t := \int_0^t \phi(s) \, dW(s)$, where $\phi$ is an $L^p$-stochastically integrable process with values in $E$.

The idea to use decoupling inequalities to construct a theory of stochastic integration in UMD spaces is due to McConnell [5] who used convergence in probability rather than $L^p$-convergence. McConnell first generalized Garling’s inequalities to obtain decoupling inequalities for tangent sequences with values in UMD spaces and used these to prove that a progressive process with values in a UMD space $E$ is stochastically integrable if and only if its trajectories are stochastically integrable almost surely as $E$-valued functions. His arguments depend heavily on the equivalence of the UMD property and the geometric notion of $\zeta$-convexity. By using stopping time arguments, our Theorem 2 can be localized to recover McConnell’s result under somewhat weaker measurability assumptions. An advantage of this approach is that it uses the UMD property in a direct and elementary way through Garling’s inequality. An Itô formula is obtained as well.

Our results can be extended to processes with values in $\mathcal{L}(H, E)$, where $H$ is a separable real Hilbert space and $E$ is a real UMD space; the integrator is then an $H$-cylindrical Brownian motion. In a subsequent paper we shall apply the results to the study of existence, uniqueness, and regularity of certain classes of nonlinear stochastic evolution equations in $E$, thereby extending parts of the theory of stochastic evolution equations in Hilbert spaces developed by Da Prato and Zabczyk [2] and many others, to the setting of UMD spaces.

**References**


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