ON CARDINALITY BOUNDS FOR HOMOGENEOUS SPACES AND THE $G_\kappa$-MODIFICATION OF A SPACE

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Abstract. Improving a result in [9], we construct a closing-off argument showing that the Lindelöf degree of the $G_\kappa$-modification of a space $X$ is at most $2^{L(X)F(X)} \kappa$, where $F(X)$ is the supremum of the lengths of all free sequences in $X$ and $\kappa$ is an infinite cardinal. From this general result follow two corollaries: (1) $|X| \leq 2^{L(X)F(X)} \text{pct}(X)$ for any power homogeneous Hausdorff space $X$, where $\text{pct}(X)$ is the point-wise compactness type of $X$, and (2) $|X| \leq 2^{L(X)F(X)} \psi(X)$ for any Hausdorff space $X$, as shown recently by Juhász and Spadaro [23]. By considering the Lindelöf degree of the related $G_{\kappa_c}$-modification of a space $X$, we also obtain two consequences: (1) if $X$ is a power homogeneous Hausdorff space then $|X| \leq 2^{aL_c(X)t(X) \text{pct}(X)}$, where $aL_c(X)$ is the almost Lindelöf degree with respect to closed sets, and (2) $|X| \leq 2^{aL_c(X)t(X) \psi_c(X)}$ for any Hausdorff space $X$, a well-known result of Bella and Cammaroto [5]. This demonstrates that both the Juhász-Spadaro and Bella-Cammaroto cardinality bounds for Hausdorff spaces are consequences of more general results that additionally lead to companion bounds for power homogeneous Hausdorff spaces. Finally, we give cardinality bounds for $\theta$-homogeneous spaces that generalize those for homogeneous spaces, including cases in which the Hausdorff condition is relaxed.

1. Introduction.

For a space $X$ and an infinite cardinal $\kappa$, the $G_\kappa$-modification of $X$, denoted by $X_\kappa$, is the space formed on $X$ with a topology generated by the $G_\kappa$-sets of $X$. Recent investigations by Arhangel'skiĭ [4] and Carlson and Ridderbos [9] have shown that certain cardinality bounds for Hausdorff spaces as well as power homogeneous Hausdorff spaces can be derived using the $G_\kappa$-modification of a space. Recall a space $X$ is homogeneous if for every $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$. $X$ is power homogeneous if $X^\kappa$ is homogeneous for some cardinal $\kappa$. By generalizing a result of Pytkeev [20], a closing-off argument was given in Theorem 3.5 in [9] showing that $L(X_\kappa) \leq 2^{L(X)t(X)\kappa}$ for any space $X$. From this follows (a) the well-known cardinality bound $2^{L(X)t(X)\psi(X)}$ for a Hausdorff space $X$ (see, for example, [16, 2.27]), and (b) the cardinality bound $2^{L(X)t(X)\text{pct}(X)}$ for a power homogeneous Hausdorff space $X$, where $\text{pct}(X)$ is the point-compactness type of $X$ defined in 2.9 below. This latter bound is a generalization of De la Vega's Theorem [10], which states that the cardinality of a compact homogeneous Hausdorff space is at most $2^{t(X)}$. 

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In this study we construct closing-off arguments that give additional bounds for the Lindelöf degree of the $G_\kappa$-modification of a space and for the related $G_\kappa^c$-modification, defined in Definition 3.3. From each result follows both a new cardinality bound for power homogeneous Hausdorff spaces and a known cardinality bound for general Hausdorff spaces. This further demonstrates the central role of the $G_\kappa$-modification $X_\kappa$ of a space $X$ in establishing cardinality bounds. Indeed we see that such bounds follow from more general results involving $X_\kappa$.

In Section 2 we establish in Theorem 2.7 that $L(X_\kappa) \leq 2^{L(X)F(X)}\kappa$ for any Hausdorff space $X$ and infinite cardinal $\kappa$, where $F(X) = \sup\{|F| : F$ is a free sequence in $X\}$. As $F(X) \leq L(X)t(X)$ for any space $X$, this improves the cardinal inequality $L(X_\kappa) \leq 2^{L(X)t(X)}\kappa$ given in [9, Theorem 3.5]. The notion of the $\kappa$-closure of a set plays the role of the tightness $t(X)$ in the proof of Theorem 2.7, allowing for the elimination of $t(X)$ altogether throughout the argument. The only proof where a free sequence is constructed is in the proof of Lemma 2.5, although this lemma is used in several places in the overall closing-off argument. From Theorem 2.7 follows (a) the recently proven cardinality bound $2^{L(X)F(X)\psi(X)}$ for Hausdorff spaces given independently by Juhász and Spadaro (see [23]), and (b) the new cardinality bound $2^{L(X)F(X)pct(X)}$ for power homogeneous Hausdorff spaces. Again, we see a cardinality bound for Hausdorff spaces and a companion bound for power homogeneous spaces following from a more general result involving the $G_\kappa$-modification.

In Section 3 we define the related $G_\kappa^c$-modification $X_\kappa^c$ of a space $X$, a modification well-suited for working with the cardinal invariant $aL_c(X)$. This is the almost Lindelöf degree with respect to closed sets, defined in Definition 2.3. It is clear that $aL_c(X) \leq L(X)$ and that equality holds if $X$ is regular. In Theorem 3.6, we modify the closing-off argument given in [9] to establish that $L(X_\kappa^c) \leq 2^{aL_c(X)t(X)}\kappa$ for any space $X$. Both the new cardinality bound $2^{aL_c(X)t(X)pct(X)}$ for a power homogeneous Hausdorff space $X$ and the well-known cardinality bound $2^{aL_c(X)t(X)\psi(X)}$ for general Hausdorff spaces given by Bella and Cammaroto [5] are corollaries. The cardinal invariant $nw_\theta(X)$ is used in Theorem 3.6, defined in 3.1, and appears to be new in the literature. The importance of $nw_\theta(X)$ is evident in the straightforward Lemma 3.2, as well as in Lemma 3.4. It is clear that $nw_\theta(X) = nw(X)$ for regular spaces. The use of $nw_\theta(X)$ allows for a fundamental re-working of the input lemmas given in [9] and results in a more “streamlined” approach.

In Section 4 we extend several results involving cardinality bounds on Hausdorff homogeneous or power homogeneous spaces to Hausdorff spaces that are $\theta$-homogeneous or power $\theta$-homogenous. These spaces are defined in 4.2. Every homogeneous space is $\theta$-homogeneous and so $\theta$-homogeneity is a weaker condition to impose on a space. Every regular $\theta$-homogeneous space is homogeneous, so a discussion of $\theta$-homogeneity is only meaningful in the context of non-regular spaces.
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In Section 5 we extend two cardinality bounds for $θ$-homogeneous Hausdorff (or Urysohn) spaces to $θ$-homogeneous spaces that may not be Hausdorff (or Urysohn). In particular we utilize the Hausdorff number $H(X)$ and the Urysohn number $U(X)$ of a space $X$, defined in 5.1. Two bounds for Hausdorff $θ$-homogeneous spaces remain valid if the Hausdorff (respectively, Urysohn) number is finite. We refer the reader to [6], where Bonanzinga, Cammaroto, and Matveev give an excellent discussion and new results concerning the Urysohn number $U(X)$ of a space $X$.

In general we do not assume any separation axiom conditions on a topological space, unless mentioned. Additionally, we refer the reader to [12] for definitions of all cardinal invariants not explicitly defined in this paper.

2. THE $G_κ$-MODIFICATION AND FREE SEQUENCES

We aim to show that if $X_κ$ is the $G_κ$-modification of a Hausdorff space $X$, then $$L(X_κ) \leq 2^{κ·L(X)} F(X),$$ where $F(X)$ is the supremum of the lengths of all free sequences in $X$. One corollary is the Juhasz-Spadaro bound $2^{L(X)} F(X) ψ(X)$ for the cardinality of any power homogeneous Hausdorff space $X$. We make a few definitions.

Definition 2.1. For a cardinal $κ$ and a space $X$, a subset $\{x_α : α \leq κ\} \subseteq X$ is a free sequence of length $κ$ if for every $β < κ$
$$cl_X \{x_α : α < β\} \cap cl_X \{x_α : α ≥ β\} = \emptyset.$$

Definition 2.2. For a cardinal $κ$, a space $X$, and $A \subseteq X$, we define the $κ$-closure operator on $A$ as follows:
$$cl_κ(A) = \bigcup_{B \in [A]^{≤κ}} cl B.$$

It is immediate that $cl_κ(cl_κ A) = cl_κ A \subseteq cl_ι(X) A = cl_X A$. This operator “$cl_κ$” will play the role of “tightness”.

Definition 2.3. For a space $X$, the cardinal invariant $aL(X)$, the almost Lindelöf degree of $X$, is the least infinite cardinal $κ$ such that for every open cover $𝒰$ of $X$ there exists a subfamily $𝒰' \subseteq 𝒰$ such that $X = \bigcup \{cl C : C \in 𝒰'\}$. The invariant $aL_c(X)$, the almost Lindelöf degree with respect to closed sets, is the smallest infinite cardinal $κ$ such that for every closed subset $C$ of $X$ and every collection $𝒰$ of open sets in $X$ that cover $C$, there is a subcollection $𝒰_0$ of $𝒰$ such that $|𝒰_0| ≤ κ$ and $\{cl U : U \in 𝒰_0\}$ covers $C$.

It is straightforward to show that $aL(X) ≤ aL_c(X) ≤ L(X)$ for any space $X$ and that $aL(X) = aL_c(X) = L(X)$ if $X$ is regular. For $A \subseteq X$, it follows that $aL(cl_ι(X) A) = aL(cl_X A) ≤ aL_c(X)$ and $L(cl_ι(X) A) = L(cl_X A) ≤ L_c(X)$. 


We use the following series of input lemmas in the main closing-off argument given in the proof of Theorem 2.7. The first is a variation on Lemma 3.3 in [9].

**Lemma 2.4.** Let X be a Hausdorff space, κ a cardinal, and Y ⊆ X such that X = clκ(Y). Then

1. nw(X) ≤ |Y|^κ-L(X), and
2. |X| ≤ |Y|^κ-ψc(X), hence nw(X) ≤ |Y|^κ-ψc(X).

**Proof.** For (a), consider the collection B = {X|clE : E ∈ [Y]≤κ}. One sees that B is a ψ-base for X and that |B| ≤ |Y|^κ. Thus ψc(X) ≤ |Y|^κ. Now use the fact that nw(X) ≤ ψc(X)^L(X), see for example 2.3b in [16].

For (b), for p ∈ X, let A_p ∈ [Y]≤κ such that p ∈ clX A_p, {U_α : α < ψc(X)} such that {p} = ∩{clXU_α : α < ψc(X)}, and B_p = {U_α ∩ A_p : α < ψc(X)}. As B_p ⊆ [Y]≤κ and |B_p| ≤ ψc(X), B_p ∈ [Y]≤κ≤ψc(X). Also, {p} = ∩{clX(U_α ∩ A_p) : α < ψc(X)}. Thus, the function X → [Y]≤κ≤ψc(X) : p → B_p is one-to-one and |X| ≤ |Y|^κ-ψc(f(X)). □

One of the key steps in the closing-off argument is to use an increasing chain of closed sets to obtain an upper bound for the space. One method of extending the closing-off argument is to replace closed sets by κ-closed sets. The following lemma starts this replacement process and will be the only fact regarding F(X) that we will need in our closing-off argument, and represents an important interaction between free sequences and the κ-closure operator.

**Lemma 2.5.** Let X be a space, B ⊆ X, and κ a cardinal such that L(X) ≤ κ. If L(clκB) > κ, then F(X) > κ.

**Proof.** Let ℓ be a cover of clκB by sets open in X and with no κ-subcover. Note that if A ⊆ clκB and |A| ≤ κ, then clX A = clκA ⊆ clκB and L(clκA) ≤ κ. Thus, there is a subcover ℓ_0 ⊆ ℓ such that |ℓ_0| ≤ κ, clκA ⊆ ∪ ℓ_0, and clκB\∩ ∪ ℓ_0 ≠ ∅. Using this fact, we can construct, by induction, {x_α}_{α < κ^+} ⊆ clκB and {ℓ_α}_{α < κ^+} such that ℓ_α ⊆ ℓ_β ⊆ ℓ for all α ≤ β < κ^+, |ℓ_α| ≤ κ, clX{x_α : α < β} ⊆ ∪ {ℓ_α : α < β} and {x_α : α ≥ β} ⊆ clκB\∪ ℓ_β. Since X\∪ ℓ_β is closed, we see that for all β < κ^+

clX{x_α : α < β} ∩ clX{x_α : α ≥ β} = ∅.

This shows that {x_α : α < κ^+} is a free sequence of length κ^+ and F(X) > κ. □

**Lemma 2.6.** Suppose X is a Hausdorff space, κ is a cardinal such that L(X)F(X) ≤ κ, Y ⊆ X, and W ⊆ Y are such that Y = clκ W and |W| ≤ 2κ. Suppose further that there exists a cover ℓ of Y consisting of Gκ-subsets of X. Then there exists a family ℓ′ ⊆ ℓ such that ℓ′ covers Y and |ℓ′| ≤ 2κ.

**Proof.** First, by Lemma 2.5, note that L(Y) ≤ κ. By Lemma 2.4, nw(Y) ≤ |W|^κ-L(Y) ≤ 2κ-L(Y) ≤ 2κ. Now, since nw(Y_κ) ≤ (nw(Y))^κ, we have nw(Y_κ) ≤ 2κ. It should now be clear how to choose the collection ℓ′. □
Theorem 2.7. \( L(X, \kappa) \leq 2^{\kappa^{L(X)F(X)}} \) for any Hausdorff space \( X \).

Proof. We assume without loss of generality that \( \kappa \geq L(X)F(X) \) and fix a cover \( \mathcal{G} \) of \( X \) which consists of \( G_{\kappa} \)-subsets of \( X \). For every \( G \in \mathcal{G} \), we fix a collection of open subsets \( \mathcal{U}(G) \) of \( X \) such that \( G = \bigcap \mathcal{U}(G) \) and of course \( |\mathcal{U}(G)| \leq \kappa \). Whenever \( \mathcal{G}' \subseteq \mathcal{G} \), then \( \mathcal{U}(\mathcal{G}') = \bigcup \{ \mathcal{U}(G) : G \in \mathcal{G}' \} \).

We will build an increasing sequence \( \{ F_\alpha : \alpha < \kappa^+ \} \) of \( \kappa \)-closed subsets of \( X \) and an increasing chain \( \{ \mathcal{G}_\alpha : \alpha < \kappa^+ \} \) of subsets of \( \mathcal{G} \) such that

1. \( |\mathcal{G}_\alpha| \leq 2^{\kappa} \) and there exists \( W_\alpha \subseteq X \) such that \( F_\alpha = c_{\kappa}W_\alpha \) and \( |W_\alpha| \leq 2^{\kappa} \).
2. \( \mathcal{G}_\alpha \) covers \( F_\alpha \).
3. Whenever \( \forall \mathcal{V} \in \{ \mathcal{U}(\mathcal{G}_\alpha) \}^{\leq \kappa} \) is such that \( X \setminus \bigcup \mathcal{V} \neq \emptyset \), then \( F_{\alpha+1} \setminus \bigcup \mathcal{V} \neq \emptyset \).

For limit ordinals \( \beta < \kappa^+ \), we let \( F_\beta \) be the \( \kappa \)-closure of \( \bigcup_{\alpha < \beta} F_\alpha \). Letting \( W_\beta = \bigcup_{\alpha < \beta} W_\alpha \), we see that \( F_\beta = c_{\kappa}W_\beta \) and \( |W_\beta| \leq 2^{\kappa} \). Then by Lemma 2.6 we may pick \( \mathcal{G}_\beta \) as required.

For successor ordinals \( \beta + 1 \), we first make sure that the closure condition is satisfied. This will add at most \( 2^{\kappa} \)-many points to \( F_\beta \) (since \( |\mathcal{U}(\mathcal{G}_\alpha)|^{\leq \kappa} \leq 2^{\kappa} \)), so by taking the \( \kappa \)-closure to obtain \( F_{\beta+1} \), we still have that there exists \( W_{\beta+1} \subseteq X \) such that \( F_{\beta+1} = c_{\kappa}W_{\beta+1} \) and \( |W_{\beta+1}| \leq 2^{\kappa} \). As before, using Lemma 2.6, we may pick \( \mathcal{G}_{\beta+1} \) as required.

Having completed the construction, we let \( \mathcal{G}' = \bigcup_{\alpha < \kappa^+} \mathcal{G}_\alpha \) and \( F = \bigcup_{\alpha < \kappa^+} F_\alpha \). It suffices to show that \( \mathcal{G}' \) covers \( X \). So suppose not and pick a point \( x \in X \setminus \bigcup \mathcal{G}' \). Let us note here that the set \( F \) is \( \kappa \)-closed and furthermore, \( \mathcal{G}' \) covers \( F \). For every \( G \in \mathcal{G}' \), we pick \( U(G) \in \mathcal{U}(G) \) such that \( x \notin U(G) \). This is possible since for such a \( G \), \( x \notin G \). Then \( \mathcal{U} = \{ U(G) : G \in \mathcal{G}' \} \) forms a cover of \( F \). Now, since \( F \) is \( \kappa \)-closed, it follows by Proposition 2.5 that \( L(F) \leq \kappa \). Thus we may find \( \forall \mathcal{V} \in \{ \mathcal{U} \}^{\leq \kappa} \) such that \( F \subseteq \bigcup \mathcal{V} \). This contradicts the closing-off condition since \( \forall \mathcal{V} \in \{ \mathcal{U}(\mathcal{G}_\alpha) \}^{\leq \kappa} \) for some \( \alpha < \kappa^+ \) and \( x \in X \setminus \bigcup \mathcal{V} \).

Corollary 2.8 (Juhasz-Spadaro, 2009). If \( X \) is Hausdorff then

\[
|X| \leq 2^{L(X)F(X)\psi(X)}.
\]

Proof. Observe that the modification \( X_{\psi(X)} \) is discrete. Thus, by Theorem 2.7,

\[
|X| = |X_{\psi(X)}| = L(X_{\psi(X)}) \leq 2^{L(X)F(X)\psi(X)}.
\]

We move now towards using Theorem 2.7 to establish a bound for the cardinality of power homogeneous Hausdorff spaces. We use the cardinal invariant \( \text{pct}(X) \), defined as follows.

Definition 2.9. For a space \( X \), the point-compactness type, \( \text{pct}(X) \), is the least cardinal \( \kappa \) such that \( X \) can be covered by compact sets \( K \subseteq X \) such that \( \chi(K, X) \leq \kappa \).
Observe that $\text{pct}(X) \leq \chi(X)$. In fact, straightforward arguments show that $\text{pct}(X)\psi(X) = \chi(X)$. We observe the following lemmas concerning $\text{pct}(X)$.

**Lemma 2.10.** For a Hausdorff space $X$, $\pi\chi(X) \leq F(X)\text{pct}(X)$. 

**Proof.** Let $\kappa = F(X)\text{pct}(X)$ and pick $p \in X$. There is a compact set $K \subseteq X$ such that $p \in K$ and $\chi(K,X) \leq \kappa$. As $K$ is compact it follows that $\pi\chi(p,K) \leq t(K) = F(K)$. (See, for example, 3.12 and 3.14(a) in [16]). As $K$ is a closed set, one easily sees that $F(K) \leq F(X)$. Hence, $\pi\chi(p,K) \leq F(X) \leq \kappa$. By 2.4.3 in [22], we have that $\pi\chi(p,X) \leq \pi\chi(p,K)\chi(K,X) \leq \kappa \cdot \kappa = \kappa$. Therefore $\pi\chi(X) \leq \kappa$, establishing the desired result. \[\Box\]

**Lemma 2.11.** For a Hausdorff space $X$, there is a nonempty compact subset $K \subseteq X$ and a set $H \subseteq X$ such that $\chi(K,X) \leq F(X)\text{pct}(X)$, $K \subseteq \text{cl}_X H$ and $|H| \leq F(X)$. 

**Proof.** Let $\kappa = F(X)\text{pct}(X)$. As $\text{pct}(X) \leq \kappa$, there exists a nonempty compact space $E \subseteq X$ such that $\chi(E,X) \leq \kappa$. Since $E$ is compact, by Theorem 2.2.4 in [2] there exists a set $K$ closed in $E$ (hence compact) and a subset $H \subseteq K$ such that $|H| \leq t(E) = F(E) \leq F(X)$, $K \subseteq \text{cl}_E H \subseteq \text{cl}_X H$, and $\chi(K,E) \leq t(E) = F(E) \leq F(X) \leq \kappa$. Now, $\chi(K,X) \leq \chi(K,E)\chi(E,X)$ as $E$ and $K$ are compact. (See, for example, 2.4.3 in [22]). Then $\chi(K,X) \leq \kappa \cdot \kappa = \kappa$. \[\Box\]

**Lemma 2.12.** If $X$ is a Hausdorff space and $K \subseteq X$ is a compact set such that $\chi(K,X) \leq \kappa$, then $K$ is a $G_{\kappa}$-set. 

**Proof.** There exists a collection $\mathcal{U}$ of open sets of $X$ such that $|\mathcal{U}| \leq \kappa$ and if $K \subseteq W$ for an open set $W$ then there exists $U \in \mathcal{U}$ such that $K \subseteq U \subseteq W$. Let $x \notin K$. Since $X$ is Hausdorff and $K$ is compact, there exists and open set $V$ such that $K \subseteq V$ and $x \notin V$. There exists $U \in \mathcal{U}$ such that $K \subseteq U \subseteq V$. This means $x \notin \bigcup \mathcal{U}$ and hence $K = \bigcap \mathcal{U} = \bigcap_{U \in \mathcal{U}} U$. \[\Box\]

**Theorem 2.13.** For a power homogeneous Hausdorff space $X$,

$$|X| \leq 2^{L(X)F(X)\text{pct}(X)}.$$ 

**Proof.** Let $\kappa = L(X)F(X)\text{pct}(X)$. By Lemma 2.11 and 2.12, there exists a non-empty $G_{\kappa}$-set $K$ contained in the closure of a set of cardinality at most $\kappa$. Now, $\pi\chi(X) \leq F(X)\text{pct}(X) \leq \kappa$ by Lemma 2.10, so by Corollary 2.9 in [3] we may find a cover $\mathcal{G}$ of $X$ consisting of $G_{\kappa}$-subsets of $X$ such that each member of $\mathcal{G}$ is contained in the closure of a subset of $X$ of cardinality at most $\kappa$. By Theorem 2.7 we may assume that $|\mathcal{G}| \leq 2^\kappa$. It follows that the density of $X$ does not exceed $2^\kappa$. Since $X$ is power homogeneous, we have that $|X| \leq d(X)^{\pi\chi(X)}$ as shown by Ridderbos in [21]. Thus, $|X| \leq d(X)^{\pi\chi(X)} \leq 2^{\kappa \cdot \pi\chi(X)} = 2^\kappa$, noting again that $\pi\chi(X) \leq F(X)\text{pct}(X) \leq \kappa$. \[\Box\]
Theorem 2.13 represents an improvement over Theorem 3.11 in [9]. Observe in the proof that power homogeneity is used twice, first through Corollary 2.9 in [3] and second through the use of the cardinality bound $d(X)^{πχ(X)}$ as shown by Ridderbos [21].

**Corollary 2.14.** For a power homogeneous Hausdorff space $X$,

$$w(X) ≤ 2^{L(F(X))pct(X)}.$$  

**Proof.** Let $κ = L(F(X))pct(X)$. By Theorem 2.13, we have that $nw(X) ≤ |X| ≤ 2^κ$. As $pct(X) ≤ κ$, by Lemma 3.4.8 in [22] it follows that $w(X) ≤ 2^κ$. □

### 3. The $G^κ_ϕ$-modification and the cardinal invariant $aL^c_ϕ(X)$

In this section we consider for a space $X$ the cardinal invariant $aL^c_ϕ(X)$, the almost Lindelöf degree with respect to closed sets, defined in Definition 2.3. We establish a general result giving cardinality bounds involving this invariant. By modifying and improving the closing-off argument given in section 3 of [9], we give a new bound for power homogeneous Hausdorff spaces using $aL^c_ϕ(X)$. The main result in this section also gives a proof of the Bella-Cammaroto [5] bound $2^{aL^c_ϕ(X)ψ_c(X)}$ for the cardinality of any Hausdorff space $X$, where $ψ_c(X)$ is the closed pseudo-character of $X$. We begin with the following definition, which seems to be new in the literature.

**Definition 3.1.** For a space $X$, a collection $N$ of non-empty subsets of $X$ is a $θ$-network for $X$ if for all $x ∈ U$, where $U$ is open, there exists $N ∈ N$ such that $x ∈ N ⊆ U$. The $θ$-network weight, $nw_θ(X)$, is the least cardinality of a $θ$-network for $X$.

Clearly $nw_θ(X) = nw(X)$ if $X$ is regular. The following lemma is a modification of Lemma 3.3 in [9] and illustrates an important relationship between the $θ$-network weight and the tightness of a space.

**Lemma 3.2.** For any space $X$, $nw_θ(X) ≤ d(X)^{t(X)}$.

**Proof.** Let $D$ be a dense subset of $X$ such that $|D| = d(X)$. Define

$$B = \{E : E ∈ [D]^{≤ t(X)}\}.$$  

Note that $|B| ≤ d(X)^{t(X)}$. We show that $B$ is a $θ$-network for $X$. Pick $x ∈ X$ and suppose $x ∈ U$ where $U$ is open. Since $x ∈ D$, there exists $F ∈ [D]^{≤ t(X)}$ such that $x ∈ F$. It is straightforward to see that $x ∈ U \cap F ⊆ U$. Also, since $U \cap F ∈ [D]^{≤ t(X)}$, we see that $U \cap F ∈ B$. □

In §2, we used the $G_κ$-modification of a space, where $κ$ is a cardinal. Here we utilize a slightly different modification, which we call the $G^c_κ$-modification of a space.
Lemma 3.4. For a space $G \subseteq X$ is a $G^{\kappa}_c$-set if there exists a collection of open sets $\mathcal{U}$ of cardinality at most $\kappa$ such that 

$$G = \bigcap \mathcal{U} = \bigcap_{U \in \mathcal{U}} U.$$ 

A $G^{\kappa}_c$ set is a special closed $G_\kappa$-set, and not every closed $G_\kappa$-set is a $G^{\kappa}_c$-set. Let $\mathcal{B}$ be the collection of $G^{\kappa}_c$-sets of $X$. Define the $G^{\kappa}_c$-modification of a space $X$, denoted by $X^{\kappa}_c$, to be space formed on $X$ with topology generated by $\mathcal{B}$.

Lemma 3.5. For a space $X$ and a cardinal $\kappa$, $nw(X^{\kappa}_c) \leq nw_\theta(X)^\kappa$.

Proof. Let $\mathcal{N}$ be a $\theta$-network for $X$ such that $|\mathcal{N}| = nw_\theta(X)$. Define 

$$\mathcal{C} = \left\{ \bigcap \mathcal{M} : \mathcal{M} \in [\mathcal{N}]^{\leq \kappa} \right\}.$$ 

Note $|\mathcal{C}| \leq nw_\theta(X)^\kappa$. We show $\mathcal{C}$ is a network for $X^{\kappa}_c$. Let $x \in G$, where $G$ is a $G^{\kappa}_c$-set of $X$. There exists a collection of open sets $\mathcal{U}$ of $X$ such that $|\mathcal{U}| \leq \kappa$ and $G = \bigcap \mathcal{U} = \bigcap_{U \in \mathcal{U}} U$. Now, $x \in U$ for all $U \in \mathcal{U}$ and there exists $N_U \in \mathcal{N}$ such that $x \in N_U \subseteq U$. Hence, $x \in \bigcap_{U \in \mathcal{U}} N_U \subseteq \bigcap_{U \in \mathcal{U}} U = G$. This shows $\mathcal{C}$ is a network for $X^{\kappa}_c$. \hfill \Box

The following lemma is a modification of Corollary 3.4 in [9].

Lemma 3.6. Let $Y$ be a closed subset of a space $X$ with $d(Y) \leq 2^\kappa$ for a cardinal $\kappa$, and let $\mathcal{G}$ be a cover of $Y$ which consists of $G^{\kappa}_c$ subsets of $X$. Then there is some family $\mathcal{G}' \subseteq \mathcal{G}$ such that $\mathcal{G}'$ covers $Y$ and $|\mathcal{G}'| \leq 2^{\kappa \cdot t(X)}$.

Proof. By Lemma 3.2, $nw_\theta(Y) \leq d(Y)^{t(Y)} \leq d(Y)^{t(X)} \leq 2^{\kappa \cdot t(X)}$ (as $Y$ is a closed subset of $X$). By Lemma 3.4, $nw(Y^{\kappa}_c) \leq nw_\theta(Y)^\kappa \leq 2^{\kappa \cdot t(X)}$. It should now be clear how to choose the collection $\mathcal{G}'$. \hfill \Box

We now modify the closing-off argument in Theorem 3.5 in [9]. Notice the changes to the closing-off condition (3).

Theorem 3.7. $L(X^{\kappa}_c) \leq 2^{2^\kappa - a L_c(X) t(X)}$ for any space $X$.

Proof. We assume without loss of generality that $\kappa \geq a L_c(X) t(X)$ and fix a cover $\mathcal{G}$ of $X$ which consists of $G^{\kappa}_c$-subsets of $X$. For every $G \in \mathcal{G}$, we fix a collection of open subsets $\mathcal{U}(G)$ of $X$ such that $G = \bigcap \mathcal{U}(G) = \bigcap \{U : U \in \mathcal{U}(G)\}$ and of course $|\mathcal{U}(G)| \leq \kappa$. Whenever $\mathcal{G}' \subseteq \mathcal{G}$, then $\mathcal{U}(\mathcal{G}') = \bigcup \{\mathcal{U}(G) : G \in \mathcal{G}'\}$.

We will build an increasing sequence $\{F_\alpha : \alpha < \kappa^+\}$ of closed subsets of $X$ and an increasing chain $\{\mathcal{G}_\alpha : \alpha < \kappa^+\}$ of subsets of $\mathcal{G}$ such that

1. $|\mathcal{G}_\alpha| \leq 2^\kappa$ and $d(F_\alpha) \leq 2^\kappa$.
2. $\mathcal{G}_\alpha$ covers $F_\alpha$.
3. Whenever $\mathcal{V} \in [\mathcal{U}(\mathcal{G}_\alpha)]^{\leq \kappa}$ is such that $X \setminus \bigcup \{\overline{V} : V \in \mathcal{V}\} \neq \emptyset$, then $F_{\alpha + 1} \setminus \bigcup \{\overline{V} : V \in \mathcal{V}\} \neq \emptyset$. 


For limit ordinals $\beta < \kappa^+$, we let $F_\beta$ be the closure of $\bigcup_{\alpha < \beta} F_\alpha$. Then $d(F_\beta) \leq 2^\kappa$ so by Lemma 3.5 we may pick $\mathcal{G}_\beta$ as required.

For successor ordinals $\beta + 1$, we first make sure that the closure condition is satisfied. This will add at most $2^\kappa$-many points to $F_\beta$ (since $|\mathcal{U}(\mathcal{G}_\beta)| \leq 2^\kappa$), so by taking the closure to obtain $F_{\beta + 1}$, we still have that $d(F_{\beta + 1}) \leq 2^\kappa$. As before, using Lemma 3.5, we may pick $\mathcal{G}_{\beta + 1}$ as required.

Having completed the construction, we let $\mathcal{G}' = \bigcup_{\alpha < \kappa^+} \mathcal{G}_\alpha$ and $F = \bigcup_{\alpha < \kappa^+} F_\alpha$. It suffices to show that $\mathcal{G}'$ covers $X$. So suppose not and pick a point $x \in X \setminus \bigcup \mathcal{G}'$. Let us note here that since $t(X) \leq \kappa$, the set $F$ is closed and furthermore, $\mathcal{G}'$ covers $F$. Now, since $x \notin F$ for every $G \in \mathcal{G}'$, and each such $G$ is of the form $G = \bigcap \mathcal{U}(G) = \bigcap \{U : U \in \mathcal{U}(G)\}$, for each $G \in \mathcal{G}'$ we pick $U(G) \in \mathcal{U}(G)$ such that $x \notin \overline{U(G)}$. Then $\mathcal{U} = \{U(G) : G \in \mathcal{G}'\}$ forms a cover of $F$, since $\mathcal{G}'$ covers $F$, and for each $G \in \mathcal{G}'$, we have $G = \bigcap \mathcal{U}(G)$ and $U(G) \in \mathcal{U}(G)$. So as $aL_c(X) \leq \kappa$ and $F$ is closed, we may find $V \in [\mathcal{U}]^{\leq \kappa}$ such that $F \subseteq \bigcup \{V : V \in \mathcal{V}\}$. We have $\mathcal{V} \subseteq [\mathcal{U}(\mathcal{G}_\alpha)]^{\leq \kappa}$ for some $\alpha < \kappa^+$. Since $x \in X \setminus \bigcup \{V : V \in \mathcal{V}\}$, by the closing-off condition (3) we have $F_{\alpha + 1} \setminus \bigcup \{V : V \in \mathcal{V}\} \neq \emptyset$. But this is a contradiction since $F_{\alpha + 1} \subseteq F \subseteq \bigcup \{V : V \in \mathcal{V}\}$. \quad \square

An immediate consequence of Theorem 3.6 is the well-known bound given by Bella-Cammaroto [5] for the cardinality of any Hausdorff space. Note, however, that no separation axioms on $X$ are assumed in Theorem 3.6.

**Corollary 3.7** (Bella-Cammaroto). For any Hausdorff space $X$, $|X| \leq 2^{aL_c(X)t(X)\psi_c(X)}$.

**Proof.** Observe that the modification $X_{\psi_c(X)}^c$ is discrete as $X$ is Hausdorff. Thus, by Theorem 3.6,

$$|X| = |X_{\psi_c(X)}^c| = L(X_{\psi_c(X)}^c) \leq 2^{aL_c(X)t(X)\psi_c(X)}.$$ \quad \square

We focus our attention now on using Theorem 3.6 to give a new cardinality bound for power homogeneous Hausdorff spaces.

In [9, Corollary 3.11], it was shown that if $X$ is a power homogeneous Hausdorff space then $|X| \leq 2^{L(X)t(X)\text{pct}(X)}$. We show that if $X$ is power homogeneous and Hausdorff, then the Lindelöf degree $L(X)$ can be replaced by $aL_c(X)$ in this bound.

**Lemma 3.8.** If $X$ is a space and $t(X)\text{pct}(X) \leq \kappa$, then there exists a non-empty compact set $K \subseteq X$ and a set $H \subseteq X$ such that $\chi(K, X) \leq \kappa$, $K \subseteq \mathcal{P}X$, and $|H| \leq \kappa$.

**Proof.** Since $\text{pct}(X) \leq \kappa$, there exists a non-empty compact set $E \subseteq X$ such that $\chi(E, X) \leq \kappa$. Since $E$ is compact, by Theorem 2.2.4 in [2] there exists a set $K$ closed in $E$ (hence compact) and a subset $H$ such that $|H| \leq t(E) \leq t(X) \leq \kappa$, $K \subseteq cl_E H \subseteq cl_X H$, and $\chi(K, E) \leq t(E) \leq \kappa$. Now
\( \chi(K, X) \leq \chi(K, E) \chi(E, X) \) since \( E \) and \( K \) are compact. (See, for example Lemma 2.4.3 in [22]). Then \( \chi(K, X) \leq \kappa \cdot \kappa = \kappa \).

\[ \]

The following is a slight variation of [3, Corollary 2.9]. Its proof is virtually identical and works not only for closed \( G_\kappa \)-sets but also for (necessarily closed) \( G_c^\kappa \)-sets.

**Proposition 3.9.** Let \( X \) be a power homogeneous Hausdorff space such that \( \pi\chi(X) \leq \kappa \) for a cardinal \( \kappa \). Suppose there exists a non-empty \( G_c^\kappa \)-set contained in the closure of a set of cardinality at most \( \kappa \). Then every point of \( X \) is contained in a \( G_c^\kappa \)-set contained in the closure of a set of cardinality at most \( \kappa \).

**Theorem 3.10.** If \( X \) is power homogeneous and Hausdorff, then
\[
|X| \leq 2^{aL_c(X)t(X)\text{pct}(X)}.
\]

**Proof.** Let \( \kappa = aL_c(X)t(X)\text{pct}(X) \). Note that the proof of Lemma 2.12 shows that if \( X \) is a Hausdorff space and \( K \subseteq X \) such that \( \chi(K, X) \leq \kappa \), then \( K \) is in fact a \( G_c^\kappa \)-set, not just a \( G_\kappa \)-set. Thus, by Lemmas 3.8 and 2.12, there exists a non-empty \( G_c^\kappa \)-set \( K \) contained in the closure of a set of cardinality at most \( \kappa \). Now, \( \pi\chi(X) \leq t(X)\text{pct}(X) \leq \kappa \) (see [22, Corollary 2.4.4], for example) so by Proposition 3.9, we may find a cover \( \mathcal{G} \) of \( X \) consisting of \( G_c^\kappa \) subsets of \( X \) such that each member of \( \mathcal{G} \) is contained in the closure of a subset of \( X \) of cardinality at most \( \kappa \). By Theorem 3.6 we may assume that \( |\mathcal{G}| \leq 2^\kappa \). It follows that the density of \( X \) does not exceed \( 2^\kappa \). Since \( X \) is power homogeneous, we have that \( |X| \leq d(X)\pi\chi(X) \) as shown by Ridderbos in [21]. Thus,
\[
|X| \leq d(X)^\pi\chi(X) \leq 2^{\kappa \cdot \pi\chi(X)} = 2^\kappa,
\]
noting again that \( \pi\chi(X) \leq t(X)\text{pct}(X) \leq \kappa \).

The proof of Theorem 3.10 is similar in structure to that of the proof of Theorem 2.13. Power homogeneity is used twice again, first though the use of Proposition 3.9, and second through the cardinality bound \( |X| \leq d(X)^\pi\chi(X) \).

**Corollary 3.11.** If \( X \) is power homogeneous and Hausdorff, then
\[
w(X) \leq 2^{aL_c(X)t(X)\text{pct}(X)}.
\]

**Proof.** Let \( \kappa = aL_c(X)F(X)\text{pct}(X) \). By Theorem 3.10, we have that \( nw(X) \leq |X| \leq 2^\kappa \). As \( \text{pct}(X) \leq \kappa \), by Lemma 3.4.8 in [22] it follows that \( w(X) \leq 2^\kappa \).

4. \( \theta \)-HOMOGENEITY AND SEMIREGULARIZATION

**Definition 4.1.** Let \( X \) and \( Y \) be spaces. For \( A \subseteq X \), we define the \( \theta \)-closure of \( A \) by
\[
cl_\theta(A) = \{ x \in X : (cl V) \cap A \neq \emptyset \text{ for every open set } V \text{ containing } X \}.
\]
A set $A \subseteq X$ is \(\theta\)-dense if $\text{cl}_\theta A = X$. We define the \(\theta\)-density $d_\theta(X)$ to be the least cardinality of a \(\theta\)-dense subset of $X$.

**Definition 4.2.** Let $X$ and $Y$ be spaces. A function $f : X \to Y$ is \(\theta\)-continuous at $x \in X$ if for each open set $V$ containing $f(x)$ there is an open set $U$ containing $x$ such that if $f[\text{cl}_X U] \subseteq \text{cl}_Y V$, $f$ is \(\theta\)-continuous if $f$ is \(\theta\)-continuous at each point $x \in X$. A bijection $f : X \to Y$ is a \(\theta\)-homeomorphism if there exists a \(\theta\)-homeomorphism $h : X \to X$ such that $h(x) = y$. $X$ is power \(\theta\)-homogeneous if there exists a cardinal $\kappa$ such that $X^\kappa$ is \(\theta\)-homogeneous.

As any homeomorphism on a space is also a \(\theta\)-homeomorphism, we see that every homogeneous space is clearly \(\theta\)-homogeneous. It is also not hard to see that a \(\theta\)-homeomorphism on a regular space is a homeomorphism, rendering a study of \(\theta\)-homogeneity meaningful only in the context of non-regular spaces. In this section we show that several known cardinality bounds for homogeneous and power homogeneous spaces hold under the weaker assumption of \(\theta\)-homogeneity or power \(\theta\)-homogeneity. For some of these results we need to use the semiregularization of a space $X$.

**Definition 4.3.** Let $X$ be a space and let $U$ be an open set of $X$. $U$ is regular open if $U = \text{int}(\text{cl}U)$. A subset $C \subseteq X$ is regular closed if $C$ is the closure of an open set. Let $\text{RO}(X)$ denote collection of regular open sets of $X$. The semiregularization $X_s$ of $X$ is the space with underlying set $X$ with $\text{RO}(X)$ as a basis. $X$ is semiregular if $X \approx X_s$. The notation $X_s$ for the semiregularization of $X$ should not cause confusion with the notation $X_\kappa$ for the $G_\kappa$-modification of a space as $s$ with will never denote a cardinal in this paper.

For a function $h : X \to X$, it is straightforward to verify that $h$ is \(\theta\)-homeomorphism if and only if $h : X_s \to X_s$ is a homeomorphism. This gives the following important proposition:

**Proposition 4.4.** A space $X$ is \(\theta\)-homogeneous if and only if $X_s$ is homogeneous.

The product of semiregular spaces is semiregular if and only if each coordinate space is semiregular. (See, for example, [19, 2.2j]). Thus, for a space $X$ and a cardinal $\kappa$, it follows that $(X^\kappa)_s \approx (X_s)^\kappa$. From this we have the following.

**Proposition 4.5.** A space $X$ is power \(\theta\)-homogeneous if and only if $X_s$ is power homogeneous.

Using semiregularization, several known cardinality bounds for power homogeneous spaces can be extended to power \(\theta\)-homogeneous spaces in a straightforward way, as the following theorem demonstrates.

**Theorem 4.6.** Let $X$ be a power \(\theta\)-homogeneous space.
(1) If $X$ is Hausdorff then
(a) $|X| \leq d(X)^{\pi\chi(X)}$, and
(b) $|X| \leq 2c(X)^{\pi\chi(X)}$.

(2) If $X$ is Urysohn then $|X| \leq d_\theta(X)^{\pi\chi(X)}$.

Proof. For a power homogeneous Hausdorff space $X$, the bound in (1)(a) was proved in [21] and that in (1)(b) was shown in [8]. If $X$ is $\theta$-power homogeneous, then $X_s$ is power homogeneous by Proposition 4.4. Now use an argument almost identical to that at the end of the proof of Theorem 5.4, noting that $c(X) = c(X_s)$ for (1)(b) as shown in [7, Lemma 2.2]. Additionally, for a power homogeneous Urysohn space $X$, the bound in (2) was shown in [7]. If $X$ is $\theta$-power homogeneous and Urysohn, then again $X_s$ is power homogeneous and proceed with an argument similar to that at the end of the proof of Theorem 5.4. □

We aim now to show that the bound given in Theorem 2.13 works if the space is $\theta$-homogeneous and Hausdorff. For this, we use the following proposition which shows that the image of a $G^c_\kappa$-set under a $\theta$-homeomorphism is still at least a $G_\kappa$-set.

**Proposition 4.7.** Let $X$ be a space and $\kappa$ a cardinal. If $h : X \to X$ is a $\theta$-homeomorphism and $K$ is a $G^c_\kappa$-set, then $h[K]$ is a $G_\kappa$-set.

Proof. As $K$ is a $G^c_\kappa$-set there exists a collection $\mathcal{U}$ of open sets such that $|\mathcal{U}| \leq \kappa$ and

$$K = \bigcap_{U \in \mathcal{U}} = \bigcap_{U \in \mathcal{U}} U.$$

For all $U \in \mathcal{U}$ and for all $x \in h[U]$ we have $h^{-1}(x) \in U$. Since $h^{-1}$ is $\theta$-continuous there exists an open set $U_x$ containing $x$ such that $h^{-1}[U_x] \subseteq \overline{U}$. Thus $U_x \subseteq h[U]$. Since $x \in U_x \subseteq \overline{U} \subseteq h[U]$, we have

$$h[U] \subseteq \bigcup_{x \in h[U]} U_x \subseteq \bigcup_{x \in h[U]} \overline{U_x} \subseteq h[U].$$

Then,

$$h[K] = \bigcap_{U \in \mathcal{U}} h[U] \subseteq \bigcap_{U \in \mathcal{U}} \bigcup_{x \in h[U]} U_x \subseteq \bigcap_{U \in \mathcal{U}} h[U] = h[K].$$

Thus,

$$h[K] = \bigcap_{U \in \mathcal{U}} \bigcup_{x \in h[U]} U_x,$$

which is a $G_\kappa$-set. □

**Theorem 4.8.** If $X$ is a $\theta$-homogeneous Hausdorff space then

$$|X| \leq 2^{L(X)F(X)\text{pct}(X)}.$$
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Proof. Let $\kappa = L(X)F(X)pct(X)$. By Lemmas 2.11 and 2.12, there exists a non-empty $G_\kappa^c$-set $K$ contained in the closure of a set of cardinality at most $\kappa$. (Note that the compact set $K$ found in Lemma 2.12 is a $G_\kappa^c$-set, not just a $G_\kappa$-set). As $X$ is $\theta$-homogeneous, by Proposition 4.7 we can find a cover $G$ of $X$ consisting of $G_\kappa^c$-subsets of $X$ such that each member of $G$ is contained in the closure of a subset of $X$ of cardinality at most $\kappa$. By Theorem 2.7, we may assume that $|G| \leq 2^{\kappa}$. Thus, the density of $X$ does not exceed $2^{\kappa}$. Now $\pi_X(X) \leq F(X)pct(X)$ by Lemma 2.10, and $|X| \leq d(X)^{\pi_X(X)}$ by Theorem 4.6(1)(a), which establishes the desired inequality. □

Question 4.9. If $X$ is a power $\theta$-homogeneous Hausdorff space, is it true that $|X| \leq 2^{L(X)F(X)pct(X)}$?

5. Cardinality bounds involving the Hausdorff and Urysohn Numbers

In this section we generalize cardinality bounds for $\theta$-homogeneous Hausdorff spaces to $\theta$-homogeneous spaces that may not be Hausdorff. We define the Hausdorff and Urysohn numbers of a space.

Definition 5.1. For a space $X$, define the Hausdorff number $H(X)$ to be the least cardinal $\kappa$ such that for all $A \subseteq X$ such that $|A| \geq \kappa$ and for all $a \in A$ there exist open sets $U_a$ containing $a$ such that $\bigcap_{a \in A} U_a = \emptyset$. Similarly, define the Urysohn number $U(X)$ to be the least cardinal $\kappa$ such that for all $A \subseteq X$ such that $|A| \geq \kappa$ and for all $a \in A$ there exist open sets $U_a$ containing $a$ such that $\bigcap_{a \in A} \overline{U_a} = \emptyset$.

As $X$ is Hausdorff if and only if $H(X) = 2$, and $X$ is Urysohn if and only if $U(X) = 2$, $H(X)$ and $U(X)$ are cardinal functions generalizing the Hausdorff and Urysohn properties. See the preprint [6] for examples and recent generalizations of cardinality bounds for Urysohn spaces to spaces with finite Urysohn number. We begin this section by giving cardinality bounds for $\theta$-homogeneous (not necessarily Hausdorff) spaces with either finite Hausdorff number or finite Urysohn number that generalize known bounds. First we observe the following:

Proposition 5.2. Let $X$ be a space.

1. If $H(X)$ is finite then $H(X) = H(X_a)$.
2. $U(X) = U(X_a)$ (even if $U(X)$ is not finite).

Proof. (1) follows from the fact that if $U_1, U_2, \ldots, U_n$ are a finite collection of open sets of $X$, then $\bigcap_{i=1}^n U_i = \emptyset$ if and only if $\bigcap_{i=1}^n \text{int}(\overline{U_i}) = \emptyset$. (2) follows since for any open set $U$ of $X$, $\overline{U} = \text{cl}(\text{int}(\overline{U}))$.

The following is also a straightforward proposition:

Proposition 5.3. If $\kappa$ is a cardinal, $X$ and $Y$ are sets, and $f : X \to Y$ is a $\kappa$-to-one function, then $|X| \leq \kappa|Y|$. 

Theorem 5.4. Let $X$ be a $\theta$-homogeneous (not necessarily Hausdorff) space. Then,

1. If $H(X)$ is finite then $|X| \leq d(X)^{\pi\chi(X)}$.
2. If $U(X)$ is finite then $|X| \leq d_\theta(X)^{\pi\chi(X)}$.

Proof. We begin by proving (2) using a variation of the proof of [7, Theorem 4.1]. Suppose first that $X$ is homogeneous. Fix a point $p \in X$, and for every $x \in X$ let $h_x : X \to X$ be a homeomorphism such that $h_x(p) = x$. Let $\mathcal{B}$ be a local $\pi$-base at $p$ such that $|\mathcal{B}| = \pi\chi(X)$. Let $D$ be a $\theta$-dense set such that $|D| = d_\theta(X)$ and let $n = U(X)$. As $D$ is $\theta$-dense, for all $B \in \mathcal{B}$ and all $x \in X$ there exists $d(x, B) \in D \cap \overline{h_x[B]}$. Define $\phi : X \to D^\mathcal{B}$ by $\phi(x)(B) = d(x, B)$.

We show that $\phi$ is $(n - 1)$-to-one. We do this by choosing a set $A \subseteq X$ of size $n$ and showing that the functions $\{\phi(a) : a \in A\}$ are not identical functions. As $A$ has size $n$ and $U(X) = n$, for all $a \in A$ there exist open sets $U_a$ such that $\bigcap_{a \in A} \overline{U_a} = \emptyset$. Now, $\bigcap_{a \in A} \overline{h_a^{-1}[U_a]}$ is an open set containing $p$. Thus there exists $B \in \mathcal{B}$ such that $B \subseteq \bigcap_{a \in A} \overline{h_a^{-1}[U_a]}$. It follows that $\overline{h_a[B]} \subseteq \overline{U_a}$ and so $\bigcap_{a \in A} \overline{h_a[B]} = \emptyset$ since $\bigcap_{a \in A} \overline{U_a} = \emptyset$. Now suppose that $d(a, B) = d(b, B)$ for all $a, b \in B$. Call this element $z$, so that $z = d(a, B)$ for all $a \in A$. Then $z \in \bigcap_{a \in A} \overline{h_a[B]} = \emptyset$, a contradiction. Thus, not all $d(a, B)$ are identical for all $a \in A$, and so not all $\phi(a)(B)$ are identical for all $a \in A$.

This means there exist distinct $a, b \in A$ such that $\phi(a)(B) \neq \phi(b)(B)$, and hence $\phi(a) \neq \phi(b)$. Thus the functions $\{\phi(a) : a \in A\}$ are not identical functions and so $\phi$ is $(n - 1)$-to-one. By Proposition 5.3, it follows that $|X| \leq |D|^{|\mathcal{B}|} = d_\theta(X)^{\pi\chi(X)}$.

Now if $X$ is $\theta$-homogeneous, it follows by Proposition 4.4 that $X_s$ is homogeneous. Applying the above argument to $X_s$, we have

$$|X| = |X_s| \leq d_\theta(X_s)^{\pi\chi(X_s)} \leq d_\theta(X)^{\pi\chi(X)}.$$  

The last inequality above follows from [7, Corollary 2.7] and the fact that $d_\theta(X) = d_\theta(X_s)$.

To prove (1), use an argument identical to the above except use $U_a$ in place of $\overline{U_a}$ and $h_a[B]$ in place of $\overline{h_a[B]}$. \qed

A natural question here is whether the bounds given in Theorem 5.4 work if “$\theta$-homogeneous” is replaced by “power $\theta$-homogeneous”.

Question 5.5. Let $X$ be a power $\theta$-homogeneous space.

1. If $H(X)$ is finite, is it true that $|X| \leq d(X)^{\pi\chi(X)}$?
2. If $U(X)$ is finite, is it true $|X| \leq d_\theta(X)^{\pi\chi(X)}$?
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References
