ON KHINTCHINE INEQUALITIES WITH A WEIGHT

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Abstract. In this note we prove a weighted version of the Khintchine inequalities.

Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a probability space and let \((r_n)_{n \geq 1}\) be a Rademacher sequence. For a random variable \(\xi : \Omega \to \mathbb{R}\) and \(p > 0\) we write \(\|\xi\|_p = (\mathbb{E}|\xi|^p)^{1/p}\). Our main result is the following weighted version of Khintchine’s inequality. We also allow the weight to be zero on a set of positive measure.

Theorem 1. Let \(0 < p < \infty\) and let \(w \in L^q(\Omega)\) for some \(q > p\), and assume \(s := \mathbb{P}(w \neq 0) > 2/3\). Let \(\xi = \sum_{n \geq 1} r_n x_n\) with \(\sum_{n \geq 1} x_n^2 < \infty\). Then there exist constants \(C_1 := C_1(p, w) > 0\) such that

\[
C_1^{-1} \left( \sum_{n \geq 1} x_n^2 \right)^{1/2} \leq \|w\|_p \leq C_2 \left( \sum_{n \geq 1} x_n^2 \right)^{1/2}.
\]

Consequently, the \(p\)-th moments for \(0 < p < q\) are all comparable.

If \(w \equiv 1\) the result reduces the Khintchine inequalities \([4]\). The nontrivial part in (1) is the first estimate. It is somewhat remarkable since the weight is not assumed to be bounded from below. To our knowledge (1) was not known before, and potentially useful. Moreover, several experts expected the result to be false. For the proof we need a well-known \(L^0\)-version of Khintchine’s inequality. We provide the details to obtain explicit constants.

Proposition 2. For all \(a \in (0, 1)\) and for all \((x_n)_{n \geq 1}\) in \(\ell^2\), one has

\[
\mathbb{P} \left( \left| \sum_{n \geq 1} r_n x_n \right| > a \right) < \left( 1 - a^2 \right)^2 / 3 \Rightarrow \sum_{n \geq 1} |x_n|^2 \leq 1
\]

We need the Paley-Zygmund inequality (see \([2, \text{Corollary 3.3.2}]\)) which says that for a positive nonzero random variable \(\xi : \Omega \to \mathbb{R}\) and \(q \in (2, \infty)\) one has

\[
\mathbb{P}(\xi > \lambda \|\xi\|_2) \geq \left( 1 - \lambda^2 \right)^{1/2} \frac{\|\xi\|_2^2}{\|\xi\|_q^2} \quad \lambda \in [0, 1].
\]

Proof. Assume \(\sum_{n \geq 1} x_n^2 > 1\). Let \(\xi = \left| \sum_{n \geq 1} r_n x_n \right|\) and \(m := \|\xi\|_2 > 1\). Recall the following case of Khintchine’s inequality: \(\mathbb{E}\xi^4 \leq 3(\mathbb{E}\xi^2)^2\) (see \([2, \text{Section 1.3}]\)). Therefore, the Paley-Zygmund inequality implies that

\[
\mathbb{P}(\xi > a) \geq \mathbb{P}(\xi > a \|\xi\|_2) \geq (1 - a^2)^2 \frac{(\mathbb{E}\xi^2)^2}{\mathbb{E}\xi^4} \geq (1 - a^2)^2 / 3.
\]

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Proof. By scaling we can assume \( 1 - 2e^{-2+\gamma} \approx 0.517 \), where \( \gamma \) is Euler constant.

Note that for \( \eta = r_1 + r_2 \) one has \( P(\eta = 0) = 1/2 \), which shows that the lemma is close to optimal.

**Lemma 3.** Let \( \xi \) applied with inequality (see [3]) one sees that for all \( q > 2 \),
\[
P(\|\xi\|_q > 0) = P(\|\eta\|_q > 0) \geq \left[B_q^{-2}\right]^{q/(q-2)},
\]
where \( B_q = \sqrt{2\left(\Gamma((q+1)/2)\right)^{1/q}} \). An elementary calculation for \( q = 2 \) shows that \( B_2^{-2q/(q-2)} = 2e^{-2+\gamma} \) as \( q \downarrow 2 \), and this implies the result. \( \square \)

**Proof of Theorem 1.** The second estimate follows from Hölder’s inequality with \( \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \) and the unweighted Khintchine inequality with constant \( k_{r,2} \):
\[
\|w\xi\|_p \leq \|w\|_q \|\xi\|_r \leq \|w\|_q k_{r,2} \left(\sum_{n \geq 1} x_n^2\right)^{1/2},
\]
where we used \( q > p \) to ensure that \( r \in (0, \infty) \).

Next we prove the first estimate. If all the \( x_n \) are zero, there is nothing to prove. If not, then by Lemma 3 and the assumption we have \( P(\|\xi\|_q > 0) = P(\|\eta\|_q > 0) > 0 \), and therefore \( \|w\xi\|_p > 0 \). To complete the proof we can assume that \( \|w\xi\|_p = 1 \) as follows by a scaling argument. Moreover, by replacing \( w \) by \( |w| \) if necessary, we can assume that \( w \) is nonnegative.

Choose \( a \in (0,1) \) so small that \( b := (1-a^2)^2/3 > 1-s \), where \( s = P(w \neq 0) \). (For example take \( a \) such that \( b = (1-a^2)^2/3 = [(1-s) + 1/3]/2 \). Let
\[
\delta_0 = \sup\{\delta > 0 : P(w > \delta) \geq (s+1-b)/2\}.
\]
Since \( P(w > 0) = s > (s+1-b)/2 \) we have \( \delta_0 > 0 \). Clearly, \( A = \{w \geq \delta_0\} \) satisfies \( P(A) \geq (s+1-b)/2 \). It follows that for all \( t > 0 \)
\[
P(\{\|\xi\|_t > 0\} \cap A) = P(1A|\xi|_t > 0) \leq \|w\|_p E(1A|\xi|_t^p)
\leq \|w\|_p \delta_0^{-p} E(|\xi|_t^p) \leq \|w\|_p \delta_0^{-p} E(|w\xi|_t^p) = \|w\|_p \delta_0^{-p} E(|\xi|_t^p).
\]
Therefore,
\[
P(\{\|\xi\|_t > 0\}) \leq P(\{\|\xi\|_t > 0\} \cap A) + P(\Omega \setminus A) < t^{-p} \delta_0^{-p} + 1 - (s+1-b)/2.
\]
Now with \( t = \delta_0^{-1}\left(b - 1 + (s+1-b)/2\right)^{-\frac{1}{p}} \) it follows that \( P(\|\xi\|_t > 0) < b \). Let \( y_n = \frac{a x_n}{t} \) and \( \eta = \sum_{n \geq 1} r_n y_n \). Then \( P(|\eta| > a) = P(|\xi|_t > b) < b \). Therefore, Proposition 2 gives that \( \sum_{n \geq 1} y_n^2 \leq 1 \). In other words \( \sum_{n \geq 1} x_n^2 \leq \frac{t^2}{a} \) and the result follows with \( C_1 = a/t \). \( \square \)

**Remark 4.** Some possible extensions are:
(1) A more sophisticated application of the Paley-Zygmund inequality in Proposition 2 shows that in the theorem it suffices to assume that $\mathbb{P}(w \neq 0) > 1 - 2e^{-2+\gamma} \approx 0.517$. This is close to optimal as can be seen by taking $w = 1_{r_1+r_2 \neq 0}$ and $\xi = r_1 + r_2$ for which the weighted inequality (1) does not hold.

(2) The integrability condition on $w$ used for the second estimate of (1) can be improved. However, the general function space for $w$ is difficult to describe and not even rearrangement invariant (cf. [1]).

(3) With a similar technique one can obtain versions of Theorem 1 for Gaussian random variables, $q$-stable random variables, etc.

(4) The case where the $x_n$ take values in a normed space $X$ can also be considered. Then $\left( \sum_{n \geq 1} x_n^2 \right)^{1/2}$ has to be replaced by the $L^2$-norm $\|\xi\|_2$, where $\xi = \sum_{n \geq 1} r_n x_n$. Note that Lemma 3 extends to this setting, as follows by applying Lemma 3 with $\eta = \langle \xi, x^* \rangle$ for a functional $x^* \in X^*$ for which $\langle \xi, x^* \rangle$ is nonzero. Also the constants in Proposition 2 can be taken as before. This follows from the fact that also in the vector-valued setting $\|\xi\|_4 \leq 3^{1/4} \|\xi\|_2$ (see [5]).

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References


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