ON SEMI-$R$-BOUNDEDNESS AND ITS APPLICATIONS

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Abstract. $R$-Boundedness is a randomized boundedness condition for sets of operators which in recent years has found many applications in the maximal regularity theory of evolution equations, stochastic evolution equations, spectral theory and vector-valued harmonic analysis. However, in some situations additional geometric properties such as Pisier’s property ($\alpha$) are required to guaranty the $R$-boundedness of a relevant set of operators. In this paper we show that a weaker property called semi-$R$-boundedness can be used to avoid these geometric assumptions in the context of Schauder decompositions and the $H^\infty$-calculus. Furthermore, we give weaker conditions for stochastic integrability of certain convolutions.

1. Introduction

$R$-boundedness has proved to be an important tool in the theory of maximal regularity of evolution equations [38], in operator theory [21], Schauder decompositions [3, 6], vector-valued harmonic analysis [12, 17] and stochastic equations (see [28] and references therein). In particular from the above results one can see that many results for Hilbert spaces extend to the Banach space setting if one replaces uniform boundedness by $R$-boundedness. There are situations in which additional geometric assumptions such as Pisier’s property ($\alpha$) are required to guaranty the $R$-boundedness of certain sets of operators (see [18, 25]). We show that in several situations these assumptions can be avoided by using the weaker notion of semi-$R$-boundedness.

Definition 1.1. Let $X$ and $Y$ be Banach spaces. Let $(r_n)_{n \geq 1}$ be a Rademacher sequence on a probability space $(\Omega, \mathcal{A}, P)$. A collection $T \subseteq \mathcal{L}(X, Y)$ is said to be $R$-bounded if there exists a constant $M \geq 0$ such that

\[(1.1) \quad \left( \mathbb{E} \left\| \sum_{n=1}^{N} r_n T_n x_n \right\|^2 \right)^{\frac{1}{2}} \leq M \left( \mathbb{E} \left\| \sum_{n=1}^{N} r_n x_n \right\|^2 \right)^{\frac{1}{2}},\]

for all $N \geq 1$ and all sequences $(T_n)_{n=1}^{N}$ in $T$ and $(x_n)_{n=1}^{N}$ in $X$.

By the Kahane-Khintchine inequalities one can replace the $L^2(\Omega; E)$-norm in (1.1) by any $L^p(\Omega; E)$-norm as long as $p \in [1, \infty)$.

If one only considers $x_n$ of the form $x_n = a_n x$, where $a_n$ is a scalar and $x \in X$, then one obtains the weaker notion of semi-$R$-boundedness:

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Definition 1.2. A collection $\mathcal{T} \subseteq \mathcal{L}(X,Y)$ is said to be semi-$R$-bounded if there exists a constant $M \geq 0$ such that

\[(1.2) \quad \left( \mathbb{E} \left\| \sum_{n=1}^{N} r_n T_n a_n x \right\|^2 \right)^{\frac{1}{2}} \leq M \left( \sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}} \|x\|,\]

for all $N \geq 1$ and all sequences $(T_n)_{n=1}^{N}$ in $\mathcal{T}$, scalars $(a_n)_{n=1}^{N}$, and $x \in X$.

This notion has been introduced and studied in [14]. In this paper we provide several characterizations and applications of semi-$R$-boundedness. Let us note that semi-$R$-boundedness is used in [4] to compare different operator norms. The maximal function which was used to study Kato’s square root in an $L^p$-setting in [15] is also defined in terms of semi-$R$-boundedness.

In this paper we give further properties and characterizations of semi-$R$-boundedness (see Sections 2 and 4) which prepare us for our main applications.

In Section 3 we give sufficient conditions for semi-$R$-boundedness in terms of smoothness of operators. We provide semi-$R$-bounded versions of results in [16] and prove sharp results for semigroups. Applications to stochastic equations are given in Section 5. Here we apply multiplier and factorization techniques to obtain path-continuity of solutions. In Section 6 we prove that the partial sum projections in a Schauder decomposition are always semi-$R$-bounded. Under geometric constrictions on the Banach space $R$-boundedness results were obtained in [32]. Finally, in Section 7 we characterize the boundedness of the $H^\infty$-calculus in terms of semi-$R$-bounded imaginary powers. Such results were known in the Hilbert space situation and for Banach spaces with so-called property $(\alpha)$ (see [22, 23, 26, 39]). We obtain a characterization for spaces with nontrivial type and also show that the $H^\infty$-calculus itself is semi-$R$-bounded.

We will write $a \lesssim b$ if there exists a universal constant $C > 0$ such that $a \leq Cb$, and $a \asymp b$ if $a \lesssim b \lesssim a$. If we want to emphasize that $C$ depends on some parameter $t$, we write $a \lesssim_t b$ and $a \asymp_t b$.

2. Definitions and basic properties

Let $X$ and $Y$ be Banach spaces. Let $(r_n)_{n \geq 1}$ be a Rademacher sequence and $(\gamma_n)_{n \geq 1}$ be a Gaussian sequence on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $\mathcal{T} \subseteq \mathcal{L}(X,Y)$. The least constant $M$ for which (1.1) in Definition 1.1 holds is called the $R$-bound of $\mathcal{T}$, and is denoted by $R(\mathcal{T})$. Replacing the Rademacher sequence $(r_n)_{n \geq 1}$ by a Gaussian sequence $(\gamma_n)_{n \geq 1}$ one obtains the definition of $\gamma$-boundedness. The least corresponding constant is denoted by $\gamma(\mathcal{T})$ and is called the $\gamma$-bound of $\mathcal{T}$. For more details on $R$-boundedness we refer to [6, 38]. If a family $\mathcal{T} \subset \mathcal{L}(X,Y)$ is $R$-bounded then it is $\gamma$-bounded and $\gamma(\mathcal{T}) \leq R(\mathcal{T})$. If $X$ has finite cotype then $\gamma$-boundedness and $R$-boundedness of $\mathcal{T}$ are equivalent and $R(\mathcal{T}) \leq C_X \gamma(\mathcal{T})$. For details on type and cotype we refer to [9].

The least constant $M$ for which (1.2) in Definition 1.2 holds is called the semi-$R$-bound of $\mathcal{T}$ and is denoted by $R_s(\mathcal{T})$. If we replace $(r_n)_{n \geq 1}$ by $(\gamma_n)_{n \geq 1}$, the least corresponding constant $M$ is denoted by $\gamma_s(\mathcal{T})$. Both conditions imply uniform boundedness with $\sup_{T \in \mathcal{T}} \|T\| \leq \gamma_s(\mathcal{T}) \leq R_s(\mathcal{T})$. Clearly, $R$-boundedness implies semi-$R$-boundedness and $\gamma$-boundedness implies semi-$\gamma$-boundedness. Moreover, by a standard randomization argument and [9, Proposition 12.11] one can see that
semi-$R$-boundedness and semi-$\gamma$-boundedness are equivalent and
\begin{equation}
\gamma_\sigma(T) \leq R_\nu(T) \leq \sqrt{\pi/2}\gamma_\sigma(T).
\end{equation}

Note that $X = \mathcal{L}(\mathbb{K}, X)$, where we associate to each $x \in X$ the operator $a \mapsto ax$. The following trivial but useful observation will allow us to reduce questions on semi-$R$-boundedness to the well-known situation of $R$-boundedness.

**Lemma 2.1.** Let $X$ and $Y$ be Banach spaces. For a collection $T \subseteq \mathcal{L}(X, Y)$ the following assertions are equivalent:

1. $T$ is semi-$R$-bounded with $R_\nu(T) \leq M$.
2. For all $x \in X$, the set $T_x = \{Tx \in \mathcal{L}(\mathbb{K}; Y) : T \in T\}$ is $R$-bounded with $R(T_x) \leq M\|x\|$.

The following result can be proved as in [14, Proposition 2.1] where the case $X = Y$ was considered.

**Proposition 2.2.** Let $X$ be a non-zero Banach space and let $Y$ be a Banach space. The following assertions are equivalent:

1. $Y$ is of type 2.
2. Every uniformly bounded collection $T \subseteq \mathcal{L}(X, Y)$ is semi-$R$-bounded.

In particular, this result implies that there are many collections $T \subseteq \mathcal{L}(X, Y)$ which are semi-$R$-bounded but not $R$-bounded.

**Example 2.3.** Let $p \in [2, \infty)$ and let $(S(t))_{t \geq 0}$ be the translation group on $L^p(\mathbb{R})$. Then $\{S(t) : t \in [-1, 1]\}$ is semi-$R$-bounded but not $R$-bounded (cf. [16, Example 6.2] and see Example 3.5 below for related results).

The following results can all be obtained from Lemma 2.1 and the corresponding $R$-boundedness result which can be found in [6, 38]. We only state the results that we need.

**Remark 2.4.** Let $T \subseteq \mathcal{L}(X, Y)$ be a collection. The following hold:

1. If $T$ is semi-$R$-bounded, then the absolute convex hull $\text{abs co}(T)$ of $T$ is semi-$R$-bounded and $R_\nu(\text{abs co}(T)) \leq 2R_\nu(T)$
2. If $T$ is semi-$R$-bounded, then the strong closure of $T$ is semi-$R$-bounded and $R_\nu(T^\text{strong}) \leq R_\nu(T)$.
3. In the definition of semi-$R$-boundedness it suffices to take the operators $T_1, \ldots, T_N$ distinct.

A space $X$ is said to be a *Grothendieck space* (GT space) if every $T : X \to \ell^2$ is 1-summing (cf. [9] for details on summing operators). Recall that $L^1$-spaces are GT spaces.

**Proposition 2.5.** Let $X$ be a Banach space, and let $Y$ be a non-zero Banach space.

1. If $X$ has cotype 2 and is a GT space, then every semi-$R$-bounded family $T$ is $R$-bounded.
2. If every semi-$R$-bounded collection $T \subseteq \mathcal{L}(X, Y)$ is $R$-bounded, then $X$ has cotype 2.

The result in the case $X = Y$ is separable has been proved in [14, Theorem 2.2] and the more general case has been considered in [4, Proposition 3.17]. In the case $X = Y$ is a separable Banach space, a complete characterization of spaces for which
semi-$R$-boundedness and $R$-boundedness coincide has been given in [14, Theorem 2.2].

One can weaken the definition of semi-$R$-boundedness by taking $a_n = 1$ for all $n$. The next result shows that this is in fact equivalent to semi-$R$-boundedness.

**Proposition 2.6.** Let $X$ and $Y$ be Banach spaces. For a collection $T \subseteq L(X, Y)$ the following assertions are equivalent:

1. The collection $T$ is semi-$R$-bounded.
2. The collection $T$ is semi-$\gamma$-bounded.
3. There exists an $M \geq 0$ such that $(\mathbb{E} \| \sum_{n=1}^{N} r_n T_n x \|^2)^{\frac{1}{2}} \leq M \sqrt{N} \|x\|$ for all $N \geq 1$, $(T_n)^N_{n=1}$ in $T$ and $x \in X$.
4. There exists an $M \geq 0$ such that $(\mathbb{E} \| \sum_{n=1}^{N} \gamma_n T_n x \|^2)^{\frac{1}{2}} \leq M \sqrt{N} \|x\|$, for all $N \geq 1$, $(T_n)^N_{n=1}$ in $T$ and $x \in X$.

Moreover, $\gamma(T) \leq R(T) \leq \sqrt{2} \gamma(T)$, and $2^{-\frac{1}{2}} \gamma(T) \leq M_{\gamma} \leq M_T \leq R(T)$, where $M_T$ and $M_{\gamma}$ are the least constants for which statements (3) and (4) above hold.

**Proof.** For (1)⇒(2) see (2.1). (2)⇒(4) and (1)⇒(3) are trivial.

(4)⇒(2): The proof is based on an approximation argument (see [19]). We first consider the case $(a_n)^N_{n=1} \in \mathbb{R}$. By symmetry we may assume $a_n \geq 0$ for all $n$. By an approximation argument it is enough to consider positive $(a_n)^N_{n=1}$ in $\mathbb{Q}$. We can find integers $K \geq 1$ and $(p_n)^N_{n=1} \in \mathbb{N}$ such that $a_n = \frac{p_n}{K}$ for all $n$. Let $(\gamma_{nm})_{n,m \geq 1}$ be a Gaussian sequence. Since $(p_n \gamma_{nm})^N_{n=1}$ and $(\sum_{m=1}^{N} \gamma_{nm})^N_{n=1}$ are identically distributed, we have

$$E \left\| \sum_{n=1}^{N} \gamma_n T_n x \right\|^2 = \frac{1}{K^2} E \left\| \sum_{n=1}^{N} p_n \gamma_n T_n x \right\|^2 = \frac{1}{K^2} E \left\| \sum_{n=1}^{N} \sum_{m=1}^{N} \gamma_{nm} T_n x \right\|^2$$

$$\leq \frac{M^2}{K^2} E \left\| \sum_{n=1}^{N} \sum_{m=1}^{N} \gamma_{nm} x \right\|^2 = \frac{M^2}{K^2} E \left\| \sum_{n=1}^{N} p_n \gamma_n x \right\|^2 = M^2 E \left\| \sum_{n=1}^{N} \gamma_n a_n x \right\|^2.$$ 

For $(a_n)^N_{n=1} \in \mathbb{C}$ we can consider the real and imaginary part separately to obtain

$$(\mathbb{E} \| \sum_{n=1}^{N} \gamma_n T_n a_n x \|^2)^{\frac{1}{2}} \leq \sqrt{2} M \|x\| \left( \sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}}.$$ 

(3)⇒(4): The result follows from a standard central limit theorem argument. Indeed, let $(r_{nk})_{n,k \geq 1}$ be a Rademacher sequence. Let $N \geq 1$ and $(x_n)^N_{n=1}$ in $X$ be arbitrary. One has

$$E \left\| \sum_{n=1}^{N} \gamma_n T_n x \right\|^2 = \lim_{K \to \infty} \frac{1}{K} E \left\| \sum_{k=1}^{K} \sum_{n=1}^{N} r_{nk} T_n x \right\|^2$$

$$\leq M^2 \lim_{K \to \infty} \frac{1}{K} E \left\| \sum_{k=1}^{K} \sum_{n=1}^{N} r_{nk} x \right\|^2 = M^2 \sqrt{N} \|x\|.$$ 

$\square$
3. Smooth operator-valued functions

In this section we show that under type and cotype assumptions certain smooth operator-valued functions have semi-$R$-bounded range. The case of $R$-boundedness has been considered in [16, Theorem 5.1]. The smoothness below is expressed in Besov and Hölder spaces. Details on Besov spaces and other spaces can be found in [36] (see [1, 35] for the vector-valued setting). For details on type and cotype we refer to [9] and references therein.

**Theorem 3.1.** Let $X$ and $Y$ be Banach spaces. Let $p \in [1, 2]$. Assume that $Y$ has type $p$. Let $T : \mathbb{R}^d \to \mathcal{L}(X,Y)$ be strongly continuous. Let $r \in [2, \infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$ and assume that there is an $M$ such that for all $x \in X$,

$$\|Tx\|_{B^{r,p}_{\infty}(\mathbb{R}^d;Y)} \leq M\|x\|.$$

Then there exists a constant $C = C(p,Y)$ such that

(3.1) $R_{s}(\{T(t) \in \mathcal{L}(X,Y) : t \in \mathbb{R}^d\}) \leq CM$.

**Remark 3.2.**

1. Note that $B^{r,p}_{\infty}(\mathbb{R}^d;Y) \hookrightarrow BUC(\mathbb{R}^d;Y)$, so that $\{T(t) : t \in \mathbb{R}^d\}$ is always uniformly bounded.

2. Theorem 3.1 also holds if $T$ is defined on a smooth domain $D \subset \mathbb{R}^d$. This easily follows from the boundedness of the extension operator (cf. [36]).

**Proof of Theorem 3.1.** Let $x \in X$ be arbitrary. Then $\{Tx \in B^{r,p}_{\infty}(\mathbb{R}^d;\mathcal{L}(\mathbb{R}^d;Y))\}$ and [16, Theorem 5.1] implies that $\{T(t)x \in \mathcal{L}(\mathbb{R}^d;Y) : t \in \mathbb{R}^d\}$ is $R$-bounded by $C(p,Y)\|x\|$. Therefore, Lemma 2.1 gives that $\{T(t) \in \mathcal{L}(X,Y) : t \in \mathbb{R}^d\}$ is semi-$R$-bounded by $C(p,Y)$.

As a consequence we obtain that Hölder regularity of an operator-valued function implies semi-$R$-boundedness which in our situation can be proved in the same way as [11, Corollary 5.4]

**Corollary 3.3.** Let $X$ and $Y$ be Banach spaces. Let $p \in [1, 2]$. Assume that $Y$ has type $p$. Let $I = (a,b)$ with $-\infty \leq a < b \leq \infty$. Let $\alpha > 0$ be such that $\alpha > \frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. Assume $T : \mathbb{R} \to \mathcal{L}(X,Y)$ and $M$ are such that for all $x \in X$, $\|Tx\|_{L^r(I;Y)} \leq M\|x\|$ and there exists an $A$ such that

(3.2) $\|T(s+h)x - T(s)x\| \leq A|h|^\alpha(1 + |s|)^{-\alpha}\|x\|$, $s, s+h \in I$, $h \in I$, $x \in X$.

Then $\{T(t) \in \mathcal{L}(X,Y) : t \in I\}$ is semi-$R$-bounded by a constant times $A$.

Note that in the case where $I$ is bounded, the factor $(1 + |s|)^{-\alpha}$ can be omitted. The situation $p = 2$ omitted as it is covered by Proposition 2.2.

Next, we prove a result on semi-$R$-boundedness of strongly continuous semigroups restricted to real interpolation spaces. The result is sharp in the smoothness index. A similar result for $R$-boundedness has been obtained in [16, Theorem 6.1]. However, there it is unclear what happens for the sharp exponent in the smoothness index.

For details on semigroups and interpolation theory we refer to [10] and [2, 36].
Corollary 3.4. Let \((S(t))_{t \in \mathbb{R}_+}\) be a strongly continuous semigroup on a Banach space \(X\) with \(|S(t)| \leq M e^{-\omega t}\) for some \(M, \omega > 0\). Assume \(X\) has type \(p \in [1, 2]\). Let \(\alpha = \frac{1}{p} - \frac{1}{2}\) and let \(i_\alpha : (X, D(A))_{\alpha,1} \to X\) be the inclusion mapping. Then
\[
\{S(t)i_\alpha : t \in \mathbb{R}_+\} \subset \mathcal{L}((X, D(A))_{\alpha,1}, X)
\]
is semi-\(R\)-bounded.

If \(S\) is not exponentially stable, then one obtains that for all \(K > 0\) the set
\[
\{S(t)i_\alpha : t \in [0, K]\} \subset \mathcal{L}((X, D(A))_{\alpha,1}, X)
\]
is semi-\(R\)-bounded. This follows from Corollary 3.4 and a translation argument.

Proof. Let \(p \in [1, 2]\). Recall from [2, Theorem 6.7.3] that \(x \in (X, D(A))_{\alpha,1}\) if and only if \(x \in X\) and
\[
\|x\|_{\alpha,1} = \|x\| + \int_0^\infty t^{-\alpha} \sup_{0 \leq h \leq t} \|S(h)x - x\| \frac{dt}{t} < \infty.
\]
Moreover, \(\|\cdot\|_{\alpha,1}\) defines an equivalent norm on \((X, D(A))_{\alpha,1}\).

Let \(N : \mathbb{R} \to \mathcal{L}(X, D(A))_{\alpha,1}\) be given by \(N(t) = S(|t|)i_\alpha\). Let \(\frac{1}{p} = \frac{1}{p} - \frac{1}{2}\). By [33, Proposition 3.1]
\[
\|Nhx\|_{B^p_{\alpha,1}(R; X)} \lesssim \|Nhx\|_{L^p(R; X)} + \int_0^\infty t^{-\alpha} \sup_{|h| \leq t} \|N(\cdot + h)x - N(\cdot)x\|_{L^p(R; X)} \frac{dt}{t}.
\]
Since \(S\) is exponentially stable, \(\|Nhx\|_{L^p(R; X)} \lesssim \|x\|.\) For the other term, using the semigroup property we get
\[
\|N(\cdot + h)x - N(\cdot)x\|_{L^p(R; X)} \lesssim \|x\|_{\alpha,1} \sup_{|h| \leq t} \|N(h)x - x\|_{\alpha,1}.
\]
Since \(N(h) = N(-h)\) it follows that
\[
\int_0^\infty t^{-\alpha} \sup_{|h| \leq t} \|N(\cdot + h)x - N(\cdot)x\|_{L^p(R; X)} \frac{dt}{t} \lesssim \|x\|_{\alpha,1} \int_0^\infty t^{-\alpha} \sup_{0 \leq h \leq t} \|S(h)x - x\| \frac{dt}{t}.
\]
Therefore, \(\|Nhx\|_{B^p_{\alpha,1}(R; X)} \lesssim M, \omega, r \|x\|_{\alpha,1}\) and the result follows from Theorem 3.1. \(\square\)

In the next example we show that the result in Corollary 3.4 is sharp.

Example 3.5. Let \(p \in [1, 2]\). Let \((S(t))_{t \in \mathbb{R}}\) be the left-translation group on \(X = L^p(\mathbb{R})\) with generator \(A = \frac{d}{dx}\).

(1) Let \(\alpha = \frac{1}{p} - \frac{1}{2}\). Then for all \(K \in \mathbb{R}_+\),
\[
\{S(t)i_\alpha : t \in [-K, K]\} \subset \mathcal{L}(B^p_{\alpha,1}(\mathbb{R}), L^p(\mathbb{R}))
\]
is semi-\(R\)-bounded. Here \(i_\alpha : B^p_{\alpha,1}(\mathbb{R}) \to L^p(\mathbb{R})\) denotes the canonical embedding. This result follows from Corollary 3.4 and \((X, D(A))_{\alpha,1} = B^p_{\alpha,1}(\mathbb{R})\) (cf. [35, Theorems 4.2 and 4.3.3]).

(2) For \(\alpha \in [0, \frac{1}{2} - \frac{1}{2})\) and \(K = 1\), the family (3.3) is not semi-\(R\)-bounded. This follows from the proof of [16, Example 6.2]

Remark 3.6. Note that if \(\alpha > \frac{1}{p} - \frac{1}{2}\), then the family in (3.3) is even \(R\)-bounded (see [16, Example 6.2]). In general we do not know whether this extends to \(\alpha = \frac{1}{p} - \frac{1}{2}\), but if \(p = 1\) this is indeed the case. This follows from Proposition 2.5, since \(B^1_{1,1}(\mathbb{R})\) (being an \(L^1\) space) is a GT-space with cotype 2.
4. Multipliers in Gauss spaces

The next proposition is a semi-$R$-bounded version of the multiplier theorem [22, Proposition 4.11]. We present the result for the measure space $((a, b), \mu, \mathcal{B}_{(a, b)})$, where $\mu$ is the Lebesgue measure. The result is valid for more general measure spaces with the same proof, but we will need it only for intervals $(a, b)$, where $-\infty < a < b < \infty$.

Let $X$ be a Banach space and $H$ be a separable Hilbert space with orthonormal basis $(h_n)_{n \geq 1}$. Let $(\gamma_n)_{n \geq 1}$ be a real-valued sequence of independent standard Gaussian random variables. An operator $R \in \mathcal{L}(H, X)$ is called $\gamma$-radonifying if $\sum_{n \geq 1} \gamma_n R h_n$ converges to some $\xi \in L^2(\Omega; X)$. Moreover, we let $\|R\|_{\gamma(H,X)} = \|\xi\|_{L^2(\Omega; X)}$.

Let $\phi : (a, b) \to \mathcal{L}(H, X)$ be strongly measurable and such that for all $x^* \in X^*$, $\phi^* x^* \in L^2(a, b; H)$. Let $I_\phi : L^2(a, b; H) \to X$ be the (Pettis)-integral operator given by

$$I_\phi f = \int_a^b \phi(t)f(t) \, dt.$$  

We say $\phi \in \gamma(a, b; H, X)$ if $I_\phi : L^2(a, b; H) \to X$ is in $\gamma(L^2(a, b; H), X)$ (i.e. if $I_\phi$ is $\gamma$-radonifying). Note that we let $\gamma(a, b; X) = \gamma(a, b; \mathbb{R}; X)$. For details on the Gauss spaces $\gamma(a, b; X)$ and $\gamma(a, b; H, X)$ we refer to [9, 22, 29].

**Proposition 4.1.** Let $X$ and $Y$ be Banach spaces. Let $S : (a, b) \to \mathcal{L}(X, Y)$ be a strongly continuous map and let $\mathcal{S} = \{S(t) \in \mathcal{L}(X, Y) : t \in (a, b)\}$. For a constant $K \geq 0$, the following assertions are equivalent:

1. $S$ is semi-$R$-bounded with $\gamma_{\text{semi}}(S) \leq K$,
2. for all $x \in X$ and all $f \in L^2(a, b)$

$$(4.1) \quad \|f S x\|_{\gamma(a,b,Y)} \leq K \|f\|_{L^2(a,b)} \|x\|_X.$$  

It actually suffices to consider indicator functions $f$ in (4.1).

**Proof.** This follows from the Gaussian version of Lemma 2.1, [22, Proposition 4.11] and the fact that $\gamma(a, b; \mathbb{K}) = L^2(a, b; \mathbb{K})$.

**Proposition 4.2.** Let $X$ and $Y$ be Banach spaces and let $H$ be a separable Hilbert space. Let $S \subset \mathcal{L}(X, Y)$ be semi-$R$-bounded by some constant $K$. Then the set $\bar{S} \subset \mathcal{L}(\gamma(H, X), \gamma(H, Y))$ defined by

$$\bar{S} = \{S : \exists S \in S \text{ such that } \forall B \in \gamma(H, X) \text{ one has } \bar{S}(B) = SB \},$$

is semi-$R$-bounded by $K$.

**Proof.** Let $S_1, \ldots, S_N \in S$ and $a_1, \ldots, a_N \in \mathbb{R}$ be arbitrary. Then

$$\left( \mathbb{E} \left[ \sum_{n=1}^{N} r_n S_n a_n B \right]^{2} \right)^{\frac{1}{2}} = \left( \mathbb{E} \gamma \left[ \sum_{n=1}^{N} \gamma_n \sum_{m=1}^{N} r_n S_n a_n B h_m \right]^{2} \right)^{\frac{1}{2}} = \left( \mathbb{E} \gamma \left[ \sum_{n=1}^{N} r_n S_n a_n \sum_{m=1}^{N} \gamma_m B h_m \right]^{2} \right)^{\frac{1}{2}} \leq K \left( \sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}} \|B\|_{\gamma(H,X)}.$$
and the result follows from Lemma 2.1.

Let $X$ be a Banach space. Let $(r_n)_{n \geq 1}$ and $(r'_n)_{n \geq 1}$ denote two independent Rademacher sequences and let $(r_{mn})_{m,n \geq 1}$ be a double indexed Rademacher sequence. We say that $X$ has property $(\alpha^+)$ if there is a constant $C$ such that for all $(x_{mn})_{m,n=1}^{M,N}$

$$
\left\| \sum_{n=1}^{N} \sum_{m=1}^{M} r_{mn} x_{mn} \right\|_{L^2(\Omega, X)} \leq C \left\| \sum_{n=1}^{N} \sum_{m=1}^{M} r'_m r_{mn} x_{mn} \right\|_{L^2(\Omega, X)}.
$$

Property $(\alpha^-)$ is defined by the opposite inequality. Properties $(\alpha^+)$ and $(\alpha^-)$ are introduced in [31]. These properties are one sided versions of Pisier’s property $(\alpha)$ (see [34]). A space $X$ has property $(\alpha)$ if and only if it has properties $(\alpha^+)$ and $(\alpha^-)$. If $X$ is a Banach function space with finite cotype, then $X$ automatically has property $(\alpha)$. Also note that the Schatten class $S^p(\ell^2)$ has property $(\alpha^+)$ (resp. $(\alpha^-)$) if and only if $p \in [2, \infty)$ (resp. $p \in [1,2]$) (cf. [31] and references therein).

**Corollary 4.3.** Let $X$ and $Y$ be Banach spaces and let $H$ be a separable Hilbert space. Let $S : (0,T) \to \mathcal{L}(X,Y)$ be a strongly continuous map. If $Y$ has property $(\alpha^+)$ and $S$ is semi-R-bounded by some constant $K$, then for all $B \in \gamma(H,X)$ and all $f \in L^2(0,T)$,

$$
\|fSB\|_{\gamma(0,T;H,Y)} \lesssim_Y K \|f\|_{L^2(0,T)} \|B\|_{\gamma(H,X)}.
$$

**Proof.** In [31, Theorem 3.3] it is shown that under property $(\alpha^+)$ the following embedding holds:

$$
\gamma(L^2(\mathbb{R}^+), \gamma(H,Y)) \hookrightarrow \gamma(L^2(\mathbb{R}^+; H), Y)).
$$

By (4.3), Propositions 4.2 and 4.1 it follows that

$$
\|fSB\|_{\gamma(0,T;H,Y)} \lesssim_Y \|fSB\|_{\gamma(0,T;\gamma(H,Y))} \leq K \|f\|_{L^2(0,T)} \|B\|_{\gamma(H,Y)}.
$$

The following result will be important in Section 5.

**Corollary 4.4.** Let $(T(t))_{t \in \mathbb{R}^+}$ be a strongly continuous semigroup on a Banach space $X$ with $\|T(t)\| \leq M e^{-\omega t}$ for some $\omega > 0$. Assume $X$ has type $p \in [1,2]$. Let $\alpha = \frac{1}{2} - \frac{1}{p}$. The following assertions hold:

1. For all $x \in (X, D(A))_{\alpha,1}$ and all $f \in L^2(\mathbb{R}^+)$, $fT^x$ is in $\gamma(\mathbb{R}^+; X)$.
2. Let $H$ be a real separable Hilbert space. If additionally $X$ has property $(\alpha^+)$, then for all $B \in \gamma(H, (X, D(A))_{\alpha,1})$ and $f \in L^2(\mathbb{R}^+)$, $fTB \in \gamma(\mathbb{R}^+; H, X)$.

**Proof.**

1. This follows from Corollary 3.4 and Proposition 4.1.
2. This follows from Corollaries 3.4 and 4.3.

**Remark 4.5.**

1. If $T$ is not uniformly exponentially stable, then the result of Corollary 4.4 still holds on finite intervals.
2. If $X$ has type 2, then property $(\alpha^+)$ is not needed in Corollary 4.4. This follows from the embedding $L^2(\mathbb{R}^+; \gamma(H,X)) \hookrightarrow \gamma(\mathbb{R}^+; H, X)$ for spaces $X$ with type 2 (see [30, Theorem 5.1]).
5. Applications to stochastic evolution equations

Let $X$ be a real Banach space and let $H$ be a real separable Hilbert space and let $T \in (0, \infty)$. We recall the stochastic Cauchy problem from [29, Section 7],
\begin{equation}
(5.1) \quad dU(t) = AU(t) \, dt + BdW_H(t), \quad t \in [0, \infty), \quad U(0) = u_0.
\end{equation}
Here $A$ is the generator of a $C_0$-semigroup $(S(t))_{t \geq 0}$ on $X$ and $B \in L(H, X)$ is a given bounded operator, and $(W_H(t))_{t \in [0, \infty)}$ is a cylindrical Wiener process. We say that (5.1) has a solution if for all $t \in [0, T]$, $\int_0^t S(t-s)B \, dW_H(s)$ exists in $L^2(\Omega; X)$. For details we refer to [29]. Let us recall from [29] that the stochastic integral exists if and only if $t \mapsto S(t)B \in \gamma(0, T; H, X)$.

The next result gives a sufficient condition for the existence of a solution to (5.1). Moreover, the solution has a version with path-wise continuous trajectories.

Theorem 5.1. Let $X$ be a Banach space. Let $Y$ be a Banach space which is continuously embedded in $X$ and let $i : Y \to X$ denote this embedding. Let $(S(t))_{t \geq 0}$ be a strongly continuous semigroup on $X$. If there exists an $\theta \in [0, \frac{1}{2})$ such that
\begin{equation}
(5.2) \quad \mathbb{R}_t \{ t^\theta S(t)i \in L(Y, X) : t \in [0, T] \} < \infty,
\end{equation}
then the following assertions hold:

1. If $B \in L(H, Y)$ has finite rank, then the problem (5.1) has a solution $(U(t))_{t \in [0, T]}$ with continuous paths.
2. If $Y$ has property $(\alpha^+)$ and $B \in \gamma(H, Y)$, then the problem (5.1) has a solution $(U(t))_{t \in [0, T]}$ with continuous paths.

Before we turn to the proof of Theorem 5.1 we give examples for the space $Y$.

Example 5.2. The semi-$R$-boundedness assumption (5.2) is fulfilled in the following three cases:

1. If $X$ has type 2, then (5.2) holds with $Y = X$ and for all $\theta \in [0, \frac{1}{2})$. This follows from Proposition 2.2.
2. Let $p \in [1, 2)$. If $X$ has type $p$, then (5.2) holds with $Y = (X, D(A))^\frac{1}{p} - \frac{1}{2}$ and for all $\theta \in [0, \frac{1}{2})$. This follows from Corollary 3.4.
3. If $S$ is analytic, then (5.2) holds with $Y = X$ and $\theta \in [0, \frac{1}{2})$. This follows from the fact that $\frac{d}{dt}[t^\theta S(t)x] \in L^1(0, T; X)$ for all $x \in X$ in $[38, \text{Proposition 2.5}]$.

Proof of Theorem 5.1. (1): We can write $B = \sum_{n=1}^N h_n \otimes x_n$, where $(h_n)_{n=1}^N$ are orthonormal and $(x_n)_{n=1}^N$ are in $Y$. It follows from Proposition 4.1 that for all $\varepsilon \in (0, 1/2 - \theta)$,
\[
\| t \mapsto t^{-\varepsilon} S(t)B \|_{\gamma(0, T; H, X)} \leq \sum_{n=1}^N \| t \mapsto t^{-\varepsilon} S(t)x_n \|_{\gamma(0, T; H, X)} \leq K_{S, \theta, T} \sum_{n=1}^N \| x_n \| \| t \mapsto t^{-\varepsilon-\theta} \|_{L^2(0, T)} < \infty.
\]
By the Banach space version of the factorization method of [8] (cf. [27] or [37]), this implies that there exists a solution with continuous paths.

2. This follows as in (1), but this time using Corollary 4.3. \hfill \Box
Note that the family \( S \) itself in Example 5.2 (3) does not have to be semi-
\( R \)-bounded. This follows from the following counterexample.

**Example 5.3.** Let \( X = L^1(0,1) \) and consider \( A = \frac{d^2}{dx^2} \), with \( D(A) = \{ x \in W^{2,1}(0,1) : x(0) = x(1) = 0 \} \). Then \( A \) generates an analytic semigroup \( (S(t))_{t \geq 0} \).

It follows from Theorem 5.1 and Example 5.2 that (5.1) has a solution with continu-
ous paths. However, \((S(t))_{t \in [0,1]}\) is not semi-
\( R \)-bounded. Indeed, it follows from the results in [14] that there do not exist semi-
\( R \)-bounded semigroups in \( L^1(0,1) \) with the property that every \( S(t) \) for \( t > 0 \) is weakly compact. Since it is well-known that \( S(t) \) is compact for all \( t > 0 \) the result follows.

### 6. Applications to Schauder decompositions

Let \( X \) be a Banach space. A sequence of bounded linear operator \((D_n)_{n \geq 1}\) in \( \mathcal{L}(X) \) is called a Schauder decomposition of \( X \) if \( D_nD_m = 0 \) for \( n \neq m \) and for all \( x \in X \), one has \( x = \sum_{n \geq 1} D_nx \). The corresponding partial sum projections \((P_n)_{n \geq 1}\) are defined by \( P_n = \sum_{k=1}^{n} D_k \). The Schauder decomposition \((D_n)_{n \geq 1}\) is called unconditional if \( \sum_{n \geq 1} D_nx \) converges unconditionally for all \( x \in X \). Under geometric conditions (property (\( \Delta \)) or weak-(\( \alpha \))) on the space \( X \) it was shown in [32] that \((P_n)_{n \geq 1}\) is \( R \)-bounded. Below we show that without any geometric assumption one always has that \((P_n)_{n \geq 1}\) is semi-
\( R \)-bounded.

**Theorem 6.1.** Let \( X \) be a Banach space. If \((D_n)_{n \geq 1}\) is an unconditional Schauder decomposition then the corresponding partial sum projections \((P_n)_{n \geq 1}\) are semi-
\( R \)-bounded.

As a consequence it follows that the column and row projections in \( S^1 \) are at least semi-
\( R \)-bounded. In [32] it is shown that they are not \( R \)-bounded.

For the proof we need a vector-valued Stein inequality for martingales with independent and symmetric increments.

**Lemma 6.2.** Let \((S, \mathcal{F}, \mu)\) be a probability space. Let \( I \subset \mathbb{R}^+ \) be an index set which starts at 0. Let \( p \in [1, \infty) \) and let \( F \) be the set of all \( f \in L^p(S;X) \) such that \((\mathbb{E}(f|\mathcal{F}_t))_{t \in I}\) defines a martingale that starts at zero and which has symmetric and independent increments. For \( t \in I \) let \( \mathbb{E}_{\mathcal{F}_t} \in \mathcal{L}(F) \) be defined by \( \mathbb{E}_{\mathcal{F}_t} f = \mathbb{E}(f|\mathcal{F}_t) \), then for all \( f \in F \) and all choices \( t_1, \ldots, t_N \in I \) and \( a_1, \ldots, a_N \in \mathbb{K} \), one has

\[
\mathbb{E}\left[ \left\| \sum_{n=1}^{N} r_n a_n \mathbb{E}_{\mathcal{F}_{t_n}} f \right\|^2_{L^p(S;X)} \right] \leq 8 \sum_{n=1}^{N} |a_n|^2 \|f\|^2_{L^p(S;X)}.
\]

**Remark 6.3.**

1. If \( X \) has property (\( \Delta \)), then \( \{\mathbb{E}_{\mathcal{F}_t}\}_{t \in I} \) in \( \mathcal{L}(F) \) is \( R \)-bounded (see [28, Lemma 2.8]).
2. If \( X \) is a UMD space and \( p \in (1, \infty) \), then \( \{\mathbb{E}_{\mathcal{F}_t}\}_{t \in I} \) in \( \mathcal{L}(L^p(S;X)) \) is \( R \)-bounded (see [5], [6, Proposition 3.8]). It is not known whether this is true for a wider class than UMD spaces.

**Proof of Lemma 6.2.** We can assume \( t_1 \leq t_2, \ldots \leq t_N \) and let \( t_0 = 0 \). For \( 1 \leq n \leq N \), write \( d_n = \mathbb{E}_{\mathcal{F}_{t_n}} f - \mathbb{E}_{\mathcal{F}_{t_{n-1}}} f \). Then \((d_n)_{n=1}^{N}\) are independent and symmetric. Expectation with respect to \( (r_n)_{n \geq 1} \) will be denoted with \( \mathbb{E}_r \). We have

\[
\sum_{n=1}^{N} r_n \mathbb{E}_{\mathcal{F}_{t_n}} a_n f = \sum_{n=1}^{N} \sum_{m=1}^{n} a_m r_n d_m = \sum_{m=1}^{N} d_m \sum_{n=m}^{N} a_n r_n.
\]
By the Kahane contraction principle applied to \((d_m)_{m=1}^N\) and the Lévy-Octaviani inequalities (cf. [24]) applied to \((a_n r_n)_{n \leq N}\) it follows that
\[
\mathbb{E}_r \left\| \sum_{n=1}^N a_n r_n \mathbb{E}_{r_n} f \right\|_{L^p(S, X)}^2 = \mathbb{E}_r \left\| \sum_{m=1}^N d_m \sum_{n=m}^N a_n r_n \right\|_{L^p(S, X)}^2 \\
\leq 4 \mathbb{E}_r \sup_{1 \leq m \leq N} \left\| \sum_{n=m}^N a_n r_n \right\| \left\| \sum_{m=1}^N d_m \right\|_{L^p(S, X)}^2 \\
\leq 8 \sum_{n=1}^N |a_n|^2 \|f\|_{L^p(S, X)}^2.
\]

**Proof of Theorem 6.1.** We follow the arguments in [32, Corollary 6.2]. Let \((\tilde{r}_n)_{n \geq 1}\) be a Rademacher sequence on some probability space \((S, \mathcal{F}, \mu)\).

Define \(g : X \to L^2(S, X)\) by \(g(x) = \sum_{n=1}^\infty \tilde{r}_n D_n x\). Then \(g\) is well-defined and for all \(x \in X\)
\[(6.1) \quad (C^-)^{-1} \|x\| \leq \|g(x)\|_{L^2(S, X)} \leq C^+ \|x\|, \quad \text{where } C^-, C^+ > 0 \text{ are constants.}
\]

For \(n \geq 0\) let \(\mathcal{F}_n = \sigma(\tilde{r}_1, \ldots, \tilde{r}_n)\). It is clear that
\[
\mathbb{E}(g(x)|\mathcal{F}_n) = g(P_n x), \quad x \in X, \quad n \geq 1.
\]
Furthermore, the martingale \((\mathbb{E}(g(x)|\mathcal{F}_n))_{n \geq 0}\) starts at zero and has independent increments. By Lemma 6.2 and (6.1) we obtain that for all \(x \in X\) and \(a_1, \ldots, a_N \in \mathbb{K},
\[
\left( \mathbb{E} \left\| \sum_{n=1}^N r_n P_n a_n x \right\|_{L^2(S, X)}^2 \right)^{\frac{1}{2}} \leq C^- \left( \mathbb{E} \left\| \sum_{n=1}^N r_n g(P_n a_n x) \right\|_{L^2(S, X)}^2 \right)^{\frac{1}{2}} \\
= C^- \left( \mathbb{E} \left\| \sum_{n=1}^N r_n a_n \mathbb{E}(g(x)|\mathcal{F}_n) \right\|_{L^2(S, X)}^2 \right)^{\frac{1}{2}} \\
\leq 2\sqrt{2} C^- \left( \mathbb{E} \left\| \sum_{n=1}^N r_n a_n g(x) \right\|_{L^2(S, X)}^2 \right)^{\frac{1}{2}} \\
\leq 2\sqrt{2} C^- C^+ \left( \mathbb{E} \left\| \sum_{n=1}^N r_n a_n x \right\|_{L^2(S, X)}^2 \right)^{\frac{1}{2}}.
\]

**7. Applications to the \(H^\infty\)-calculus**

Let \(X\) be a Banach space. For details on the \(H^\infty\)-calculus we refer the reader to [13, 21, 23]. We briefly recall the definition here.

For \(\sigma \in [0, \pi)\), let \(\Sigma_\sigma = \{ \lambda \in \mathbb{C} : \arg(\lambda) < \sigma, \lambda \neq 0 \}\). As usual \(\partial \Sigma_\sigma\) will be orientated counterclockwise. Let \(H^\infty(\Sigma_\sigma)\) denote the space of bounded analytic functions \(f : \Sigma_\sigma \to \mathbb{C}\) with norm \(\|f\|_{H^\infty(\Sigma_\sigma)} = \sup_{\lambda \in \Sigma_\sigma} |f(\lambda)|\). Let
\[
H^\infty_0(\Sigma_\sigma) = \left\{ f \in H^\infty(\Sigma_\sigma) : \exists \epsilon > 0 \text{ s.t. } |f(\lambda)| \leq \frac{|z|^{\sigma}}{(1 + |z|^2)^{\epsilon}} \right\}.
\]
We say that a closed densely defined operator $A$ on a Banach space $X$ is a sectorial operator of type $w \in [0, \pi)$ if $A$ is one to one with dense range, and for all $\sigma \in (w, \pi)$ and for all $\lambda \in \Sigma_\sigma$, $\|AR(\lambda, A)\| \leq C_\sigma$.

Let $A$ be a sectorial operator of type $w \in [0, \pi)$ and fix $\sigma \in (w, \pi)$ and $\nu \in (w, \sigma)$. For $f \in H^\infty_0(\Sigma_\sigma)$ we can define

$$f(A) = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} f(\lambda) R(\lambda, A) \, d\lambda,$$

where the integral converges in the Bochner sense. We say that $A$ has a bounded $H^\infty(\Sigma_\sigma)$-calculus if there is a constant $C$ such that

$$\|f(A)\| \leq C \|f\|_{H^\infty(\Sigma_\sigma)} \text{ for all } f \in H^\infty_0(\Sigma_\sigma).$$

In this case (7.1) has a unique continuous extension to all $f \in H^\infty(\Sigma_\sigma)$.

Recall the following result:

**Theorem 7.1** ([26]). A sectorial operator $A$ on a Hilbert space $X$ has a bounded $H^\infty$-calculus if and only if it has bounded imaginary powers.

This result does not extend to the Banach space setting (see [7, Section 5]) in the sense that there exist sectorial operators with bounded imaginary powers which do not have an $H^\infty$-calculus.

If one replaces the assumption that $A$ has bounded imaginary powers by the stronger assumption that $A$ has $R$-bounded imaginary powers, then this implies again that $A$ has a bounded $H^\infty$-calculus. This is proved in [22] (also see [23, Corollary 12.11]). For spaces $X$ with property $(\alpha)$ the boundedness of the $H^\infty$-calculus is characterized by $R$-bounded imaginary powers (see [22, 23]). Below we prove a characterization of the boundedness of the $H^\infty$-calculus in terms of semi-$R$-bounded imaginary powers. Here we do not assume that $X$ has property $(\alpha)$, but require that $X$ has nontrivial type. There are many spaces (i.e. $S^p(\ell^2)$ with $p \in (1, \infty)$) which have nontrivial type but fail property $(\alpha)$.

**Theorem 7.2.** Let $X$ be a Banach space. Let $A$ be a sectorial operator of type $w$. Let $\sigma_1, \sigma_2, \sigma_3 \in [w, \pi)$. Consider the following assertions.

(a) $A$ has a bounded $H^\infty(\Sigma_{\sigma_1})$-calculus.

(b) The families

$$T_1 = \{f(A) : \|f\|_{H^\infty(\Sigma_{\sigma_2})} \leq 1\} \subset \mathcal{L}(X)$$

and $T_1^* \subset \mathcal{L}(X^*)$ are both semi-$R$-bounded.

(c) The families

$$T_2 = \{e^{-\sigma_3|t|}A^t : t \in \mathbb{R}\} \subset \mathcal{L}(X)$$

and $T_2^* \subset \mathcal{L}(X^*)$ are both semi-$R$-bounded.

The following implications hold:

1. Assume $X$ has non-trivial type. If $\sigma_2 > \sigma_1$, then (a) $\Rightarrow$ (b).
2. If $\sigma_3 \geq \sigma_2$, then (b) $\Rightarrow$ (c).
3. If $\sigma_1 > \sigma_3$, then (c) $\Rightarrow$ (a).

Roughly, the theorem can be rephrased as follows: If $X$ has nontrivial type, then the boundedness of the $H^\infty$-calculus is equivalent to semi-$R$-bounded imaginary powers.
Proof. (1): Fix \( \nu \in (\sigma_1, \sigma_2) \). Fix \( f \in H^\infty_0(\Sigma_{\sigma_2}) \). Note that by [21, Proposition 4.2] or [23, Lemma 12.4]

\begin{equation}
(7.2) \quad f(A) = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} f(\lambda) R(\lambda, A) \, d\lambda = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} f(\lambda) \lambda^{-\frac{1}{2}} A^{\frac{1}{2}} R(\lambda, A) \, d\lambda.
\end{equation}

Let \( \varphi_b \in H^\infty_0(\Sigma_{\sigma_2}) \) for \( b \in \{-1, 1\} \) be defined by \( \varphi_b(z) = z^b (e^{b \nu} - z)^{-1} \). Then by (7.2) we obtain that

\begin{align*}
(7.3) \quad f(A) &= \lim_{K \to \infty} \sum_{b \in \{-1, 1\}} -\frac{b e^{b \nu}}{2\pi i} \sum_{K}^{K} \int_{2b}^{2b+1} f(e^{b \nu} t) \varphi_b(t^{-1} A) \frac{dt}{t} \\
&= \lim_{K \to \infty} \sum_{b \in \{-1, 1\}} -\frac{b e^{b \nu}}{2\pi i} \sum_{K}^{K} \int_{1}^{2} f(2^{k} e^{b \nu} t) \varphi_b(2^{-k} t^{-1} A) \frac{dt}{t} \\
&= \lim_{K \to \infty} \sum_{b \in \{-1, 1\}} -\frac{b e^{b \nu}}{2\pi i} \int_{1}^{2} \sum_{K}^{K} f(2^{k} e^{b \nu} t) \varphi_b(2^{-k} t^{-1} A) \frac{dt}{t}.
\end{align*}

Using this we show the semi-\( R \)-boundedness of \( T_1 \). Let \( x \in X \) be arbitrary. Let \( a_1, \ldots, a_N \in \mathbb{K} \) be arbitrary. By Remark 2.4 (2) it suffices to consider \( f_1, \ldots, f_N \in H^\infty_0(\Sigma_{\sigma_2}) \) with \( \|f_n\|_{H^\infty(\Sigma_{\sigma_2})} \leq 1 \), \( n = 1, \ldots, N \). Fix \( \omega \in \Omega \). Let \( x^* \in X^* \) be such that \( \|x^*\| \leq 1 \) and

\[ \left\| \sum_{n=1}^{N} r_n(\omega) a_n f_n(A)x \right\| = \left( \sum_{n=1}^{N} r_n(\omega) a_n f_n(A)x, x^* \right). \]

Then with \( F_k : [1, 2] \times \Omega \to X \) given by \( F_k(t, \omega) = \sum_{n=1}^{N} r_n(\omega) a_n f_n(2^k e^{b \nu} t) \), it follows from (7.3) that

\begin{align*}
\left\| \sum_{n=1}^{N} r_n(\omega) a_n f_n(A)x \right\| &= \lim_{K \to \infty} \sum_{b \in \{-1, 1\}} -\frac{b e^{b \nu}}{2\pi i} \int_{1}^{2} \sum_{n=1}^{N} r_n(\omega) a_n \sum_{K}^{K} f_n(2^{k} e^{b \nu} t) \varphi_b(2^{-k} t^{-1} A)x, x^* \frac{dt}{t} \\
&= \lim_{K \to \infty} \sum_{b \in \{-1, 1\}} -\frac{b e^{b \nu}}{2\pi i} \int_{1}^{2} \sum_{K}^{K} f_n(t, \omega) \varphi_b(2^{-k} t^{-1} A)x, x^* \frac{dt}{t}.
\end{align*}

Let \((\tilde{r}_k)_{k \in \mathbb{Z}}\) be a Rademacher sequence on some probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\). Expectations with respect to \( \tilde{\Omega} \) and \( \Omega \) will be denoted by \( \tilde{E} \) and \( E \) respectively. It follows that

\begin{align*}
\left| \sum_{K}^{K} f_n(t, \omega) \varphi_b(2^{-k} t^{-1} A)x, x^* \right| &\leq \left( \tilde{E} \left\| \sum_{K}^{K} \tilde{r}_k F_n(t, \omega) \varphi_b(2^{-k} t^{-1} A)x, x^* \right\|^2 \right)^{\frac{1}{2}} \\
&\leq \left( \tilde{E} \left\| \sum_{K}^{K} \tilde{r}_k F_n(t, \omega) \varphi_b(2^{-k} t^{-1} A)x \right\|^2 \right)^{\frac{1}{2}} \left( \tilde{E} \left\| \sum_{K}^{K} \tilde{r}_k \varphi_b(2^{-k} t^{-1} A)^* x^* \right\|^2 \right)^{\frac{1}{2}}. 
\end{align*}
Recall from [23, Theorem 12.2] that

\begin{equation}
(7.4) \quad \mathbb{E} \left\| \sum_{k=-K}^{K} \tilde{r}_k \varphi_k^* (2^{-k} t^{-1} A) x \right\| \leq C_1 \|x\|,
\end{equation}

\begin{equation}
(7.5) \quad \mathbb{E} \left\| \sum_{k=-K}^{K} \tilde{r}_k \varphi_k^* (2^{-k} t^{-1} A)^* x^* \right\| \leq C_2 \|x^*\|.
\end{equation}

By integration over \( \Omega \), Fatou’s lemma, the Kahane-Khintchine inequality and (7.5) we can conclude that

\[ \mathbb{E} \left\| \sum_{n=1}^{N} r_n a_n f_n(A) x \right\| \leq C_2 \liminf_{K \to \infty} \int_1^2 \mathbb{E} \left\| \sum_{k=-K}^{K} \tilde{r}_k F_k(t, \cdot) \varphi_k^* (2^{-k} t^{-1} A) x \right\| dt / t. \]

Since \( X \) has non-trivial type, it also has some cotype \( q < \infty \) (see [9, Chapter 13]). Therefore, by [16, Lemma 3.1] or [20, Lemma 3.1] we obtain that

\[ \mathbb{E} \left\| \sum_{k=-K}^{K} \tilde{r}_k F_k(t, \cdot) \varphi_k^* (2^{-k} t^{-1} A) x \right\| \leq X,q \operatorname{sup}_{k \in \mathbb{Z}} \| F_k(t, \cdot) \|_{L^{q+1}(\Omega)} \left( \mathbb{E} \left\| \sum_{k=-K}^{K} \tilde{r}_k \varphi_k^* (2^{-k} t^{-1} A) x \right\|^2 \right)^{\frac{1}{2}} \leq \sup_{k \in \mathbb{Z}} \| F_k(t, \cdot) \|_{L^{q+1}(\Omega)}^2 \|x\|, \]

where in the last line we used (7.4) and the Kahane-Khintchine inequality. Again by the Khintchine inequality it follows that

\[ \| F_k(t, \cdot) \|_{L^{q+1}(\Omega)} \leq q \| F_k(t, \cdot) \|_{L^2(\Omega)} = \left( \sum_{n=1}^{N} |a_n|^2 |f_n(2^k e^{bi \omega t})|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}}. \]

Putting things together we obtain that

\[ \mathbb{E} \left\| \sum_{n=1}^{N} r_n a_n f_n(A) x \right\| \leq X,q C_1 C_2 \left( \sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}} \|x\| \]

which proves the semi-\( R \)-boundedness of \( T_{1} \).

The semi-\( R \)-boundedness of \( T_{1}^* \) can be proved in a similar way. Indeed, taking adjoints in (7.3) one obtains that

\[ f(A)^* = \lim_{K \to \infty} \sum_{b \in \{-1,1\}} \frac{-be^{b\nu \omega t}}{2\pi i} \int_{-K}^{K} f(2^k e^{b\nu \omega t}) \varphi_k (2^{-k} t^{-1} A)^* dt / t. \]

Fix \( x^* \in X^* \). Let \( a_1, \ldots, a_N \in \mathbb{K} \) be arbitrary. Let \( f_1, \ldots, f_N \in H_0^{\infty}(\Sigma_{\sigma_2}) \) be such that \( \|f_n\|_{H^{\infty}(\sigma_2)} \leq 1 \), \( n = 1, \ldots, N \). Fix \( \delta > 0 \). Fix \( \omega \in \Omega \). Let \( x \in X \) be such that \( \|x\| \leq 1 + \delta \) and

\[ \left\| \sum_{n=1}^{N} r_n(\omega) a_n f_n(A)^* x^* \right\| = \left\langle x, \sum_{n=1}^{N} r_n(\omega) a_n f_n(A)^* x^* \right\rangle. \]

Then it follows that

\[ \left\| \sum_{n=1}^{N} r_n(\omega) a_n f_n(A)^* x^* \right\| \]
operator of type Corollary 7.3. Let
\[\sum_{n=1}^{N} r_n a_n f_n(A)^* x^*\]
\[\lesssim (1 + \delta) C_1 \liminf_{K \to \infty} \frac{1}{t} \int_{1}^{2} E \| \sum_{k=-K}^{K} \tilde{r}_k F_k(t) \varphi_k^{\frac{1}{2}} (2^{-k} t^{-1} A)^* x^* \| \frac{dt}{t}.\]

Since $X$ has non-trivial type, $X^*$ has finite cotype. Therefore, one can complete the proof in the same way as before.

(2): This follows by taking $f_t(z) = e^{-\sigma t |z|^2}$, $t \in \mathbb{R}$.

(3): It follows from Proposition 4.1 that
\[\| t \mapsto e^{-\sigma_1 t |A^t x|} \|_{L^2(\mathbb{R}, X)} \leq C \| e^{-\sigma_1 |A^t x|} \|_{L^2(\mathbb{R})} \| x \| \lesssim_{\sigma_1, \sigma_3, C} \| x \|.\]

In the same way we obtain that
\[\| t \mapsto e^{-\sigma_1 |t (A^t)^* x^*|} \|_{L^2(\mathbb{R}, X)} \lesssim_{\sigma_1, \sigma_3, C} \| x^* \|.\]

Now the result follows from [22, Theorem 7.2].

**Corollary 7.3.** Let $X$ be a Banach space with property (a). Let $A$ be a sectorial operator of type $w$ and let $\sigma > w$. If the families
\[T(\sigma) = \{ f(A) : \| f \|_{H^\infty(\mathbb{S}^2)} \leq 1 \} \subset L(X)\]
and $T^*(\sigma) \subset L(X^*)$ are both semi-$R$-bounded, then $T(\sigma')$ is $R$-bounded for all $\sigma' > \sigma$.

**Proof.** This follows from Theorem 7.2, and [22, Corollary 7.5] or [23, Theorem 12.8].

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**References**


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